Acyclic Edge Coloring of Subdivisions of Halin Graphs

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Joint work with Ko-Wei Lih
Outline

1. Acyclic Edge Colorings
2. Acyclic Edge Coloring Conjecture
3. Main Results
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3. Main Results
Edge Colorings

5-edge coloring

1 4 4 1 1

2 2

3

4 4

5 5
All graphs mentioned in this talk are finite, without loops or parallel edges.

**Definition**

A proper edge coloring of a graph $G$ is called **acyclic** if:

- there is no 2-colored cycle in $G$,
- any cycle of $G$ is colored with at least 3 colors,
- the union of any two color classes induces a subgraph of $G$ which is a forest.

**Definition**

The **acyclic chromatic index**, denoted $a'(G)$, is the minimum $k$ such that $G$ has an acyclic $k$-edge coloring.
Acyclic Edge Colorings

5-acyclic edge coloring

Hsin-Hao Lai

Acyclic Edge Coloring of Subdivisions of Halin Graphs
A Question

Vizing Theorem (1964)
For any graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Question
Does $\Delta(G) \leq a'(G) \leq \Delta(G) + 1$ hold for any graph $G$?
A Question

Vizing Theorem (1964)

For any graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Question

Does $\Delta(G) \leq a'(G) \leq \Delta(G) + 1$ hold for any graph $G$?

No. $a'(K_{2n}) > \Delta(K_{2n}) + 1 = 2n$ if $n \geq 2$. 

![Diagram of K_{2n} graph with edge coloring]

1 2 3 4
Acyclic Edge Coloring Conjecture (AECC)

For any graph $G$, $\Delta(G) \leq a'(G) \leq \Delta(G) + 2$.

The conjecture was independently posed by Fiamčík in 1978 and Alon, Sudakov, and Zaks in 2001.
**Known Results**

\[ \Delta(G) \leq a'(G) \leq \Delta(G) + 2 \text{ if} \]

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Description</th>
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<tbody>
<tr>
<td>Fiamčík 1984</td>
<td>( G ) is a graph with ( \Delta(G) \leq 3 ).</td>
</tr>
<tr>
<td>Alon, Sudakov, Zaks 2001</td>
<td>( G ) is a graph with ( \text{girth}(G) \geq c\Delta(G) \log \Delta(G) ) for some constant ( c ).</td>
</tr>
<tr>
<td>Basavaraju, Sunil Chandran 2009</td>
<td>( G ) is connected, ( \Delta(G) \leq 4 ), and ( | G | \leq 2</td>
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Known Results of Planar Graphs

\[ \Delta(G) \leq a'(G) \leq \Delta(G) + 2 \text{ if} \]

**Sun, Wu 2008**

- \( G \) is planar, each pair of 4-cycles are edge-disjoint, and no cycles of length 3, 5.

**Borowiecki, Fiedorowicz 2009**

- \( G \) is planar and \( girth(G) \geq 5 \) or
- \( G \) is planar and contains no cycles of length 4, 6, 8, 9.

**Hou, Wu, Liu, Liu 2009**

- \( G \) is planar and \( girth(G) \geq 5 \).
Known Results of $a'(G) \leq \Delta(G) + 1$

$\Delta(G) \leq a'(G) \leq \Delta(G) + 1$, for

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<th>$G$ is a graph with $\Delta(G) \leq 3$ and $G \neq K_4, K_{3,3}$.</th>
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<tr>
<td>Něsetřil, Wormald 2005</td>
<td>almost all random $d$-regular graphs.</td>
</tr>
<tr>
<td>Muthu, Narayanan, Subramaniann 2005</td>
<td>$G$ is a partial 2-tree, an outerplanar graph, or a partial torus.</td>
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Known Results of $a'(G) \leq \Delta(G) + 1$

\[ \Delta(G) \leq a'(G) \leq \Delta(G) + 1, \text{ for} \]

**Xu, Chen, Mu 2006**

$G$ is a Halin graph and $G \neq K_4$.

**Basavaraju, Sunil Chandran 2008**

$G$ is a 2-degenerate graph.

**Hou, Wu, Liu, Liu 2009**

$G$ is planar and $\text{girth}(G) \geq 7$;
### Known Results of $a'(G) = \Delta(G)$

$a'(G) = \Delta(G)$ if

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<td>$G$ is planar, $girth(G) \geq 16$, and $\Delta(G) \geq 3$.</td>
</tr>
<tr>
<td>$G$ is an outerplanar graph and $\Delta(G) \geq 5$.</td>
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</tbody>
</table>
Main Results

- A graph operator preserving \( a'(G) \leq \Delta(G) + 1 \).
- \( \Delta(G) \leq a'(G) \leq \Delta(G) + 1 \) if \( G \) is a subdivision of a Halin graph and \( G \neq K_4 \).
- A graph operator preserving \( a'(G) = \Delta(G) \).
- \( a'(G) = \Delta(G) \) if \( G = T \cup C \) is a Halin graph, \( \Delta(G) \geq 6 \), each vertex in \( C \) belongs to a triangle, and \( G \) contains two triangles sharing a common edge.
Attaching a Cycle of Type \((2,1,1,1,2,3,1,1,4,1,2)\)
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A Graph Operator Preserving $a'(G) \leq \Delta(G) + 1$

**Theorem**

Assume that $S(G)$ is a graph obtained from $G$ by attaching a cycle of type $(l_1, l_2, \ldots, l_k)$, where $\sum_{i=1}^{k} l_i \geq 4$. Let $l_i \geq 2$ for some $i$ or let $v_{j,1}$ have no neighbor in $G$ for some $j$. If $a'(G) \leq \Delta(G) + 1$, so is $S(G)$.
If $a'(G) \leq \Delta(G) + 1$, Then $a'(S(G)) \leq \Delta(S(G)) + 1$

$l_i \geq 2$ for some $i$

adding a path
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$l_i \geq 2$ for some $i$
adding a path
vertices of degree 2
If \( a'(G) \leq \Delta(G) + 1 \), then \( a'(S(G)) \leq \Delta(S(G)) + 1 \).

\( v_{j,1} \) has no neighbor in \( G \) for some \( i \)

adding a path
If $a'(G) \leq \Delta(G) + 1$, Then $a'(S(G)) \leq \Delta(S(G)) + 1$

$\nu_{j,1}$ has no neighbor in $G$ for some $i$

adding a path
If $a'(G) \leq \Delta(G) + 1$, Then $a'(S(G)) \leq \Delta(S(G)) + 1$

$v_{j,1}$ has no neighbor in $G$ for some $i$ adding a path
If $a'(G) \leq \Delta(G) + 1$, then $a'(S(G)) \leq \Delta(S(G)) + 1$.

$\nu_{j,1}$ has no neighbor in $G$ for some $i$.

Adding a path.
A Graph Operator Preserving $a'(G) \leq \Delta(G) + 1$

Theorem

Assume that $S(G)$ is a graph obtained from $G$ by attaching a cycle of type $(l_1, l_2, \ldots, l_k)$, where $\sum_{i=1}^{k} l_i \geq 4$. Let $l_i \geq 2$ for some $i$ or let $v_{j,1}$ have no neighbor in $G$ for some $j$. If $a'(G) \leq \Delta(G) + 1$, so is $S(G)$. 
A Halin graph $H$ is a plane graph obtained by drawing a tree $T$ in the plane, where $T$ has no vertex of degree 2, then drawing a cycle $C$ through all leaves in the plane.
Theorem

If \( G \) is a subdivision of a Halin graph and \( G \neq K_4 \), then \( a'(G) \leq \Delta(G) + 1 \).
Theorem

Assume that $S(G)$ is a graph obtained from $G$ by attaching a cycle of type $(l_1, l_2, \ldots, l_k)$, where $\Sigma_{i=1}^{k} l_i \geq 4$. Let $l_i \geq 3$ for some $i$, $l_j \geq 2$ for each $j$, and $\Delta(S(G)) \geq 6$. If $a'(G) = \Delta(G)$, so is $S(G)$. 
Halin Graphs

Theorem

If $G = T \cup C$ is a Halin graph, $\Delta(G) \geq 6$, each vertex in $C$ belongs to a triangle, and $G$ contains two triangles sharing a common edge, then $a'(G) = \Delta(G)$.
# Main Results

- A graph operator preserving $a'(G) \leq \Delta(G) + 1$.
- $\Delta(G) \leq a'(G) \leq \Delta(G) + 1$ if $G$ is a subdivision of a Halin graph and $G \neq K_4$.
- A graph operator preserving $a'(G) = \Delta(G)$.
- $a'(G) = \Delta(G)$ if $G = T \cup C$ is a Halin graph, $\Delta(G) \geq 6$, each vertex in $C$ belongs to a triangle, and $G$ contains two triangles sharing a common edge.