A Unified Approach to Box-Mengerian Hypergraphs

Wenan Zang
Department of Mathematics
University of Hong Kong
Hong Kong

* Based on joint work with Xujin Chen and Zhibin Chen
§1. Introduction

- Linear system — $Ax \leq b$

- Polyhedron — $P = \{x : Ax \leq b\}$

- Integral Polyhedron — each face contains integral vectors.

- Linear Programming Duality —

\[
\begin{align*}
\text{Max } & \{c^T x \mid Ax \leq b\} \quad (1) \\
= & \text{Min } \{y^T b \mid y^T A = c^T, y \geq 0\} \quad (2)
\end{align*}
\]

**Fact 1.** $P$ is integral iff (1) has an integral optimal solution for all integral $c$ for which the optimum is finite.

**Definition 1.** Call $Ax \leq b$ totally dual integral (TDI) if (2) has an integral optimal solution for all integral $c$ for which the optimum is finite.
Definition 2. Call $Ax \leq b$ box-totally dual integral (box-TDI) if system

$$Ax \leq b; \quad \ell \leq x \leq u$$

is TDI for each pair of rational vectors $\ell$ and $u$.

Question 1. Under which conditions
(1) does a linear system define an integral polyhedron?
(2) is a linear system TDI?
(3) is a linear system box-TDI?

- Polyhedral combinatorics — application of various aspects of the theory of polyhedra and linear systems to combinatorial problems.
Fact 2. If $P = \{x : Ax \leq b\}$ is integral, then

\[
\begin{align*}
\text{Max} & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \text{ integral}
\end{align*}
\]

Theorem 1. (Edmonds & Giles ’77)

Let $Ax \leq b$ be a TDI-system. If $b$ is integral and the optimum of

\[
\begin{align*}
\text{Max} \{c^T x \mid Ax \leq b\} & = \text{Min} \{y^T b \mid y^T A = c^T, y \geq 0\}
\end{align*}
\]

is finite, then (1) has an integral optimal solution.
Corollary.

\[ \begin{align*}
\text{Max} & \quad c^T x = \text{Min} \quad y^T b \\
\text{s.t.} & \quad Ax \leq b \quad \text{s.t.} \quad y^T A = c^T \\
& \quad x \text{ integral} \quad \quad y \geq 0, \text{ integral} \\
\end{align*} \]

\[ \uparrow \]

Min-Max relation holds

- **Importance of Min-Max Relation**

(1) Combinatorial min-max relations yield elegant mathematical theorems e.g. Max-Flow Min-Cut Theorem.

(2) A min-max relation can serve as an optimality criterion, and **usually** leads to polynomial solvability of the corresponding problems.

(See A. Schrijver, Min-Max Results in Combinatorial Optimization, 1983)
§2. Complexity

Problem 1. (Schrijver, Theory of Linear and Integer Programming, '86)

Given a linear system $\pi$, how difficult is it to decide whether

(a) $\pi$ defines an integral polyhedron?

(b) $\pi$ is TDI?

(c) $\pi$ is box-TDI?

Conjecture 1. (Edmonds & Giles '84)

It is co-NP-complete to decide whether a given linear system is TDI.

Theorem 2. (Cook, Lovász & Schrijver '84)

For each fixed $r$, there exists a polytime algorithm to test if a given system $Ax \leq b$ is TDI, where $rk(A) = r$. 
**Theorem 3.** (Papadimitrou & Yannakakis '90)

It is co-NP-complete to decide whether a given linear system defines an integral polyhedron.

**Theorem 4.** (Ding, Feng & Zang; MP '08)

Let \( A \) be a \( 0 - 1 \) matrix with precisely two 1’s in each column. Then the problems of deciding whether the linear system \( Ax \geq 1, x \geq 0 \)

(a) defines an integral polyhedron;

(b) is TDI;

(c) is box-TDI

are all co-NP-complete.

**Corollary.** The above problems are all NP-hard.
§3.  Hypergraphs

• Hypergraph $\mathcal{H} = (V, \mathcal{E})$ — $V$ is finite and $\mathcal{E}$ is a family of subsets of $V$.

• $A = \text{the } \mathcal{E} - V \text{ incidence matrix.}

**Definition 3.** Call $\mathcal{H}$ **ideal** if $Ax \geq 1$, $x \geq 0$ defines an integral polyhedron.

**Definition 4.** Call $\mathcal{H}$ **Mengerian** if $Ax \geq 1$, $x \geq 0$ is TDI.

**Definition 5.** Call $\mathcal{H}$ **box-Mengerian** if $Ax \geq 1$, $x \geq 0$ is box-TDI.
• $w(v)$ — a nonnegative integral weight on each $v \in V$.

• **Edge packing** — a collection $\mathcal{C}$ of edges (repetition allowed) such that each $v$ is contained in at most $w(v)$ members of $\mathcal{C}$.

• **Edge packing problem** — to find an edge packing with max size.

• **Vertex cover** — a vertex subset $X$ that intersects all edges in $\mathcal{H}$.

• **Vertex cover problem** — to find a vertex cover with min total weight.

• $\nu_w(\mathcal{H}) = \operatorname{Max} \{|\mathcal{C}| : \mathcal{C} - \text{edge packing}\}$

• $\tau_w(\mathcal{H}) = \operatorname{Min} \{ \sum_{v \in X} w(v) : X - \text{vertex cover} \}$
**Fact 3.** $\mathcal{H}$ is **ideal** iff $\forall w \in \mathbb{Z}_+^V$ we have

$$
\tau_w(\mathcal{H}) = \text{Min}\ \{w^T x \mid Ax \geq 1, \ x \geq 0\}.
$$

**Fact 4.** $\mathcal{H}$ is **Mengerian** iff $\nu_w(\mathcal{H}) = \tau_w(\mathcal{H}) \ \forall \ w \in \mathbb{Z}_+^V$.

**Fact 5.** For a general hypergraph $\mathcal{H}$, we have $\nu_w(\mathcal{H}) \leq \tau_w(\mathcal{H})$ and the ratio can be arbitrarily large even when $w \equiv 1$.

**Fact 6.** $\mathcal{H}$ is **box-Mengerian** iff both the primal and dual of the following LP-duality equation have integral optimal solutions $\forall w, \ell, u \in \mathbb{Z}^V$.

$$
\min \{w^T x : Ax \geq 1, x \geq 0, u \geq x \geq \ell\}
$$

$$
= \max\{\alpha^T 1 + \beta^T \ell - \gamma^T u : \alpha^T A + \beta^T - \gamma^T \leq w^T, \ \alpha, \beta, \gamma \geq 0\}
$$
Theorem 5. (Schrijver '03)

Let $Ax \geq b$, $x \geq 0$ be a linear system. Suppose that for any rational vector $c$, the program $\min \{c^T x : Ax \geq b, \ x \geq 0\}$ has (if finite) an optimal dual solution $y$ such that the rows of $A$ corresponding to positive components of $y$ form a totally unimodular submatrix of $A$. Then $Ax \geq b$, $x \geq 0$ is box-TDI.

- Totally unimodular matrix — each submatrix has detem. 1, −1, or 0.
- Collection — multiset in which elements may occur more than once.
- Union of Collections $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$ — $X \cup Y = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\}$.
- $\mathcal{H} = (V, \mathcal{E})$ — hypergraph.
• \( \Lambda \) — collection of edges in \( \mathcal{H} \).

• \( d_{\Lambda}(v) \) — number of edges in \( \Lambda \) that contain \( v \).

**Definition 6.** An **equitable subpartition** of \( \Lambda \) consists of two collections \( \Lambda_1 \) and \( \Lambda_2 \) of edges in \( \mathcal{E} \) (which are not necessarily in \( \Lambda \)) such that

(i) \( |\Lambda_1| + |\Lambda_2| \geq |\Lambda| \);

(ii) \( d_{\Lambda_1 \cup \Lambda_2}(v) \leq d_{\Lambda}(v) \) for all \( v \in V \); and

(iii) \( \max\{d_{\Lambda_1}(v), d_{\Lambda_2}(v)\} \leq \lceil d_{\Lambda}(v)/2 \rceil \) for all \( v \in V \).

Call \( \mathcal{H} \) **equitably subpartitionable (ESP)**, if every collection of its edges admits an equitable subpartition. We refer to the above (i), (ii), and (iii) as **ESP property**.
**Theorem 6.** (Ding and Zang '02)

Every ESP hypergraph is Mengerian.

**Theorem 7.** (Chen, Chen and Zang '09)

Every ESP hypergraph is box-Mengerian.
• **Half-integral number** $x$ — $2x$ is an integer.

• **Half-integral vector** $y$ — $2y$ is integral.

• **Half-integral Polyhedron** — each face contains half-integral vectors.

**Definition 7.** Call $Ax \leq b$ **totally dual half-integral** (TDI/2) if (2) has an integral optimal solution for all integral $c$ for which the optimum is finite.

\[
\begin{align*}
\text{Max} & \left\{ c^T x \mid Ax \leq b \right\} \\
& = \text{Min} \left\{ y^T b \mid y^T A = c^T, y \geq 0 \right\}
\end{align*}
\]

**Definition 8.** Call $Ax \leq b$ **box-totally dual half-integral** (box-TDI/2) if system

\[Ax \leq b; \quad \ell \leq x \leq u\]

is TDI/2 for each pair of rational vectors $\ell$ and $u$.  

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• $\Lambda$ — collection of edges in a hypergraph $\mathcal{H}$.

**Definition 9.** A **quasi-equitable subpartition** of $\Lambda$ consists of two collections $\Lambda_1$ and $\Lambda_2$ of edges in $\mathcal{E}$ (which are not necessarily in $\Lambda$) such that

(i) $|\Lambda_1| + |\Lambda_2| \geq |\Lambda|$;

(ii) $d_{\Lambda_1 \cup \Lambda_2}(v) \leq d_{\Lambda}(v)$ for all $v \in V$; and

(iii') $\max\{d_{\Lambda_1}(v), d_{\Lambda_2}(v)\} \leq 2\lceil d_{\Lambda}(v)/4 \rceil$ for all $v \in V$; and

(iv) $|d_{\Lambda_1}(v) - d_{\Lambda_2}(v)| \leq 2$ for all $v \in V$ with $d_{\Lambda_1 \cup \Lambda_2}(v) = d_{\Lambda}(v)$.

Call $\mathcal{H}$ **quasi-equitably subpartitionable (QESP)**, if every collection of its edges admits an equitable subpartition. We refer to the above (i), (ii), (iii') and (iv) as **QESP property**.
• $A$ = the $E - V$ incidence matrix of a hypergraph $\mathcal{H} = (V, E)$.

**Definition 10.** Call $\mathcal{H}$ **half-ideal** if $Ax \geq 1, \ x \geq 0$ defines a half-integral polyhedron.

**Definition 11.** Call $\mathcal{H}$ **half-Mengerian** if $Ax \geq 1, \ x \geq 0$ is TDI/2.

**Definition 12.** Call $\mathcal{H}$ **box-half-Mengerian** if $Ax \geq 1, \ x \geq 0$ is box-TDI/2.

**Theorem 8.** (Chen, Chen and Zang ’09)

Every QESP hypergraph is box-half-Mengerian.

**Theorem 7.** (Chen, Chen and Zang ’09)

Every ESP hypergraph is box-Mengerian.
§4. Applications

(4.1) Vertex Covers

• $G = (V, E)$ — a graph with a nonnegative integral weight function $w$ defined on $V$.

• **Vertex cover** — a vertex subset $U$ such that $G - U$ contains no edges.

• **Vertex cover problem** — to find a vertex cover with minimum total weight.

• **Approximation** — within a factor of 2 (theorem) but not within $2 - \varepsilon$ no matter how small $\varepsilon > 0$ is (conjecture).

• $A = \text{the } E - V$ incidence matrix.
Theorem 9. (Balinski ’65)

The fractional vertex cover polyhedron \( P = \{x : Ax \geq 1, 1 \geq x \geq 0\} \) is half-ideal (that is, every vertex of \( P \) is half-integral).

Theorem 10. (Chen, Chen and Zang ’09)

Every graph is QESP.

Corollary 1.

Let \( A \) be a \( 0-1 \) matrix with precisely two 1’s in each row. Then the linear system \( Ax \geq 1, x \geq 0 \) is box-TDI/2.
**Proof of Theorem 10.** Let $G = (V, E)$ be a graph and let $\Lambda$ be an edge collection of $G$. We aim to show that $\Lambda$ admits a quasi-equitable subpartition. For this purpose, let $U$ be the set of all vertices of $G$ that are incident with some edges in $\Lambda$ and let $H = (U, \Lambda)$. WLOG, we may assume that $H$ is connected. Let $H^* = H$ if $H$ is Eulerian and let $H^*$ be obtained from $H$ by adding a new vertex $v^*$ and then making it adjacent to all vertices of odd degree in $H$ otherwise. Then $H^*$ admits an Eulerian tour $T$. Let $a$ be the starting vertex of $T$. Clearly we may assume that

(1) $a$ is precisely $v^*$, if any, and $d_{\Lambda}(a) \equiv 2 \pmod{4}$ if $H$ is Eulerian and has an odd number of edges.

Let $E_1$ consist of all odd-numbered edges in $T$ and let $E_2$ consist of all even-numbered edges in $T$. It is easy to see that $d_{E_1}(v) = d_{E_2}(v)$ for all vertices $v$ of $H^*$, except possibly vertex $a$ when $H^*$ has an odd number of edges; in this case $d_{E_1}(a) - d_{E_2}(a) = 2$. Set $\Lambda_i = \Lambda \cap E_i$ for $i = 1, 2$. Then $(\Lambda_1, \Lambda_2)$ is a quasi-equitable subpartition of $\Lambda$. \hfill \Box
Definition 9. A quasi-equitable subpartition of \( \Lambda \) consists of two collections \( \Lambda_1 \) and \( \Lambda_2 \) of edges in \( \mathcal{E} \) (which are not necessarily in \( \Lambda \)) such that

(i) \(|\Lambda_1| + |\Lambda_2| \geq |\Lambda|\);

(ii) \(d_{\Lambda_1 \cup \Lambda_2}(v) \leq d_\Lambda(v)\) for all \(v \in V\); and

(iii') \(\max\{d_{\Lambda_1}(v), d_{\Lambda_2}(v)\} \leq 2\lceil d_\Lambda(v)/4 \rceil\) for all \(v \in V\); and

(iv) \(|d_{\Lambda_1}(v) - d_{\Lambda_2}(v)| \leq 2\) for all \(v \in V\) with \(d_{\Lambda_1 \cup \Lambda_2}(v) = d_\Lambda(v)\).

Call \( \mathcal{H} \) quasi-equitably subpartitionable (QESP), if every collection of its edges admits an equitable subpartition. We refer to the above (i), (ii), (iii') and (iv) as QESP property.
(4.2) Edge Covers

- $G = (V, E)$ — a graph.

- **Edge cover** — an edge subset $M$ that covers each vertex of $G$.

- $B = \text{the } V - E \text{ incidence matrix}.$

- **Star** $S(v)$ — consisting of all edges incident with a vertex $v$.

- **Star hypergraph** $\mathcal{H} = (E, S)$ — $S$ is the set of all stars.
**Theorem 11.** (Balinski '65)

The fractional edge cover polyhedron \( Q = \{ x : Bx \geq 1, x \geq 0 \} \) is half-ideal.

**Theorem 12.** (Schrijver '03)

Let \( B \) be a 0–1 matrix with precisely two 1’s in each column. Then \( Bx \geq 2, x \geq 0 \) is TDI/2.

**Theorem 13.** (Chen, Chen and Zang '09)

Every star hypergraph is QESP.

**Corollary 2.**

Let \( B \) be a 0–1 matrix with precisely two 1’s in each column. Then the linear system \( Bx \geq 1, x \geq 0 \) is box-TDI/2.
Proof of Theorem 13. Let $\mathcal{H} = (E, S)$ be the star hypergraph of a graph $G = (V, E)$ and let $\Lambda$ be an edge collection of $\mathcal{H}$. We aim to show that $\Lambda$ admits a quasi-equitable subpartition. For convenience, we view $\Lambda$ as a star collection of $G$. Let $m_\Lambda(v)$ denote the multiplicity of $S(v)$ in $\Lambda$, let $X$ be the set of all vertices $v$ of $G$ with $m_\Lambda(v) \equiv 3 \pmod{4}$, and let $Y = V - X$. Now let $\Lambda_1$ be the star collection such that $S(v)$ appears $\lceil m_\Lambda(v)/2 \rceil$ times for any $v \in X$ and $\lfloor m_\Lambda(v)/2 \rfloor$ times for any $v \in Y$, and let $\Lambda_2$ be the star collection such that $S(v)$ appears $\lfloor m_\Lambda(v)/2 \rfloor$ times for any $v \in X$ and $\lceil m_\Lambda(v)/2 \rceil$ times for any $v \in Y$. Then $(\Lambda_1, \Lambda_2)$ is a quasi-equitable subpartition of $\Lambda$. \hfill \Box
(4.3) Cycle Hypergraphs

- $G = (V, E)$ — a graph (directed or undirected).

- **Cycle hypergraph** $\mathcal{H} = (V, \mathcal{E})$ — $\mathcal{E}$ consists of the vertex sets of all cycles in $G$.

- **Odd ring** — a graph obtained from an odd cycle by replacing each edge $e = xy$ with either a triangle containing $e$ or two triangles $xab$, $ycd$ together with two additional edges $ac$ and $bd$ (see Figure 1).

![Figure 1: An odd ring obtained from a cycle of length 7.](image)
• **Wheel** — obtained from a cycle by adding a new vertex and making it adjacent to all vertices of the cycle.

• **\( \mathcal{L} \)** — the class of all simple undirected graphs containing no induced subgraph isomorphic to a subdivision of an odd ring, or \( K_{2,3} \), or a wheel.

**Theorem 14.**

Let \( G = (V, E) \) be a simple undirected graph and let \( \mathcal{H} = (V, \mathcal{E}) \) be its cycle hypergraph. Then the following statements are equivalent:

(i) \( G \in \mathcal{L} \);

(ii) \( \mathcal{H} \) is ideal;

(iii) \( \mathcal{H} \) is Mengerian;

(iv) \( \mathcal{H} \) is box-Mengerian; and

(v) \( \mathcal{H} \) is ESP.
Remark. The equivalence of (i), (ii), (iii), and (v) was established by Ding and Zang (JCTB '02); our contribution here is to strengthen the original total dual integrality as box-total dual integrality.

Figure 2: Two forbidden subgraphs
Theorem 15.

Let $G = (V, E)$ be a tournament and let $\mathcal{H} = (V, E)$ be its cycle hypergraph. Then the following statements are equivalent:

(i) $G$ contains neither $F_1$ nor $F_2$ as a subgraph (see Figure 2);

(ii) $\mathcal{H}$ is ideal;

(iii) $\mathcal{H}$ is Mengerian;

(iv) $\mathcal{H}$ is box-Mengerian; and

(v) $\mathcal{H}$ is ESP.

Remark. The equivalence of (i) and (iii) was derived by Cai, Deng and Zang (SICOMP '01).
(4.4) Matroid Ports

- $M$ — a matroid on $E \cup \{\ell\}$, where $\ell \notin E$ is a distinguished element.

- $\mathcal{E} = \{P : P \subseteq E \text{ with } P \cup \{\ell\} \text{ a circuit of } M\}$.

- $\ell$-port — hypergraph $\mathcal{P}_{M,\ell} = (E, \mathcal{E})$.

**Theorem 16.** (Seymour ’77; Fulkerson Prize ’79)

The hypergraph $\mathcal{P}_{M,\ell}$ is Mengerian iff $M$ has no $U_{2,4}$-minor using $\ell$, and has no $F_7^*$-minor using $\ell$.

- $\mathcal{S}$ — the set of all pairs $(M, \ell)$, such that $M$ has no $U_{2,4}$-minor using $\ell$, no $F_7^*$-minor using $\ell$, and no $F_7^+$-minor using $\ell$ as a series element.

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Theorem 17.

Let $M$ be a matroid on $E \cup \{\ell\}$ with $\ell \not\in E$. Then the following statements are equivalent:

(i) $(M, \ell) \in \mathcal{S}$;

(ii) $\mathcal{P}_{M,\ell}$ is box-Mengerian; and

(iii) $\mathcal{P}_{M,\ell}$ is ESP.

Remark. The equivalence of (i) and (ii) was derived by Chen, Ding, and Zang (MOR ’08). Our proof curtails many technical parts of Chen, Ding, and Zang’s original proof and hence is much easier to follow.