Adjacent Vertex Distinguishing Edge and Total Colorings of Graphs

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I Definitions

**Edge-\(k\)-coloring:** \(\phi : E \to \{1, 2, \ldots, k\}\) such that \(\phi(x) \neq \phi(y)\) for two adjacent edges \(x\) and \(y\).

**Edge chromatic number:**
\[
\chi'(G) = \min\{k | G \text{ has an edge-}k\text{-coloring.}\}
\]
Let $C_{\phi}(v) = \{\phi(xv) \mid xv \in E\}$ denote the set of colors assigned to edges incident to the vertex $v$. 
Adjacent-Vertex-Distinguishing edge coloring (AVD): an edge coloring such that $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices $u$ and $v$.

Adjacent-Vertex-Distinguishing edge chromatic number:
\[\chi'_a(G) = \min\{k | G \text{ has a AVD edge-}k\text{-coloring}\} .\]
**Total-\(k\)-coloring:** \(\phi : V \cup E \rightarrow \{1, 2, \cdots, k\}\) such that \(\phi(x) \neq \phi(y)\) for two adjacent or incident elements \(x\) and \(y\).

**Total chromatic number:**
\[
\chi''(G) = \min\{k | G \text{ has a total-}\k\text{-coloring.}\}
\]
Let $C^*_\phi(v) = \{\phi(v)\} \cup \{\phi(xv) \mid xv \in E\}$ denote the set of colors assigned to a vertex $v$ and those edges incident to $v$. 
Adjacent-Vertex-Distinguishing total coloring (AVD): a total coloring such that $C^*_\phi(u) \neq C^*_\phi(v)$ for any pair of adjacent vertices $u$ and $v$.

Adjacent-Vertex-Distinguishing total chromatic number:
$\chi''_a(G) = \min\{k | G \text{ has a AVD total-}k\text{-coloring}\}$. 
II Some Known Results

\( \Delta: \) the maximum degree of the graph \( G \)

\( \delta: \) the minimum degree of the graph \( G \)

\( \chi(G): \) the vertex chromatic number of the graph \( G \)

\( k \)-vertex: a vertex of degree \( k \)

Normal graph: a graph without isolated edges
(2.1) **AVD edge coloring**

**Vizing Theorem.** [Vizing, 1964] For a simple graph $G$, 
$\Delta \leq \chi'(G) \leq \Delta + 1$.

**Conjecture 1.** [Zhang et al. 2002] For a normal graph $G(\neq C_5)$, $\chi'_a(G) \leq \Delta + 2$. 
If $G$ has two adjacent $\Delta$-vertices, then $\chi'_a(G) \geq \Delta + 1$.

For a normal tree $T$, $\Delta \leq \chi'_a(T) \leq \Delta + 1$; $\chi'_a(T) = \Delta + 1$ if and only if $T$ has adjacent $\Delta$-vertices.

★ If $G$ is normal and $\Delta = 3$, then $\chi'_a(G) \leq 5$.

★ If $G$ is normal and bipartite, then $\chi'_a(G) \leq \Delta + 2$.

★ If $G$ is normal, then $\chi'_a(G) \leq \Delta + O(\log \chi(G))$.

If $G$ is normal and $\Delta > 10^{20}$, then $\chi'_a(G) \leq \Delta + 300$.

(2.2) AVD total coloring

**Total Coloring Conjecture.** [Behzad 1965; Vizing 1968]

For a simple graph $G$, $\chi''(G) \leq \Delta + 2$.

**Conjecture 2.** [Zhang et al., 2004]

For a graph $G$ with $|G| \geq 2$, $\chi''_a(G) \leq \Delta + 3$. 
\[ \Delta + 1 \leq \chi''(G) \leq \chi_a''(G). \]

\[ \chi_a''(G) \leq \chi(G) + \chi'(G). \]

If \( G \) has two adjacent \( \Delta \)-vertices, then
\[ \chi_a''(G') \geq \Delta + 2. \]
If $G$ is planar, then $\chi''(G) \leq 4 + \Delta + 1 = \Delta + 5$.

(Use Four-Color Theorem and Vizing Theorem)

If $\chi'(G) = \Delta$ and $\chi(G) \leq 3$, then $\chi''(G) \leq \Delta + 3$. 
If $C_n$ is a cycle with $n \geq 3$, then $4 \leq \chi_a''(C_n) \leq 5$, and $\chi_a(C_n) = 5$, if and only if $n = 3$.

$\chi_a''(K_n) = n + 1$ if $n \equiv 0$(mod 2);
$\chi_a''(K_n) = n + 2$ if $n \equiv 1$(mod 2).

Let $n + m \geq 2$. Then $\chi_a''(K_{m,n}) = \Delta + 1$ if $m \neq n$; $\chi_a''(K_{m,n}) = \Delta + 2$ if $m = n$. 
For a tree $T$ with $|T| \geq 2$, $\chi''_a(T) \leq \Delta + 2$; $\chi''_a(T) = \Delta + 2$ if and only if $T$ has adjacent $\Delta$-vertices.


If $\Delta = 3$, then $\chi''_a(G) \leq 6$.


III   Main Results

(3.1) AVD total coloring of outerplanar graphs

A planar graph is called *outerplanar* if there is an embedding of $G$ into the Euclidean plane such that all the vertices are incident to the unbounded face.
If $G$ is an outerplanar graph, then
$$\chi(G) \leq 3, \ \chi'(G) = \Delta,$$ thus
$$\chi''_a(G) \leq \chi(G) + \chi'(G) \leq \Delta + 3.$$ 

If $G$ is a 2-connected outerplanar graph with $\Delta = 3$, then $\chi''_a(G) = 5$.

If $G$ is a 2-connected outerplanar graph with $\Delta = 6$, then $7 \leq \chi_a''(G) \leq 8$; $\chi_a''(G) = 8$ if and only if $G$ has adjacent $\Delta$-vertices.

Theorem 1. (Wang and Wang, 2008) Let $G$ be an outer-plane graph with $\Delta \geq 3$.

(1) $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$;

(2) $\chi''(G) = \Delta + 2$ if and only if $G$ has adjacent $\Delta$-vertices.
(3.2) AVD total coloring of graphs with lower maximum average degree

The *maximum average degree* $\text{mad}(G)$ of a graph $G$ is defined by

$$\text{mad}(G) = \max_{H \subseteq G} \{ 2|E(H)|/|V(H)| \}.$$
Theorem 2. (Wang and Wang, 2009) Let $G$ be a graph.

(1) If $\text{mad}(G) < 3$ and $\Delta \geq 5$, then $\Delta + 1 \leq \chi''_{a}(G) \leq \Delta + 2$; and $\chi''_{a}(G) = \Delta + 2$ if and only if $G$ contains adjacent $\Delta$-vertices.

(2) If $\text{mad}(G) < 3$ and $\Delta = 4$, then $\chi''_{a}(G) \leq 6$.

(3) If $\text{mad}(G) < \frac{8}{3}$ and $\Delta \leq 3$, then $\chi''_{a}(G) \leq 5$. 
Let $G$ be a planar graph, then

$$\text{mad}(G) < \frac{2g(G)}{g(G) - 2}.$$ 

(1) If $g(G) \geq 6$, then $\text{mad}(G) < 3$.

(2) If $g(G) \geq 8$, then $\text{mad}(G) < \frac{8}{3}$.

(3) If $g(G) \geq 10$, then $\text{mad}(G) < \frac{5}{2}$.

(4) If $g(G) \geq 14$, then $\text{mad}(G) < \frac{7}{3}$. 
Corollary 3. Let $G$ be a graph.

(1) If $g(G) \geq 6$ and $\Delta \geq 5$, then $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$; and $\chi''(G) = \Delta + 2$ if and only if $G$ contains adjacent $\Delta$-vertices.

(2) If $g(G) \geq 6$ and $\Delta = 4$, then $\chi''(G) \leq 6$.

(3) If $g(G) \geq 8$ and $\Delta = 3$, then $\chi''(G) \leq 5$. 
(3.3) AVD edge coloring of graphs with lower maximum average degree

**Theorem 4.** (Wang and Wang, 2009) Let $G$ be a graph.

1. If $\text{mad}(G) < 3$ and $\Delta \geq 3$, then $\chi'_a(G) \leq \Delta + 2$.

2. If $\text{mad}(G) < \frac{5}{2}$ and $\Delta \geq 4$, or $\text{mad}(G) < \frac{7}{3}$ and $\Delta \leq 3$, then $\chi'_a(G) \leq \Delta + 1$.

3. If $\text{mad}(G) < \frac{5}{2}$ and $\Delta \geq 5$, then $\Delta \leq \chi'_a(G) \leq \Delta + 1$ and $\chi'_a(G) = \Delta + 1$ if and only if $G$ contains adjacent $\Delta$-vertices.
Corollary 5. Let $G$ be a graph.

(1) If $g \geq 6$ and $\Delta \geq 3$, then $\chi'_a(G) \leq \Delta + 2$.

(2) If $g \geq 10$ and $\Delta \geq 4$, or $g \geq 14$ and $\Delta = 3$, then $\chi'_a(G) \leq \Delta + 1$.

(3) If $g \geq 10$ and $\Delta \geq 5$, then $\chi'_a(G) = \Delta + 1$ if and only if $G$ contains adjacent $\Delta$-vertices.
(3.4) AVD edge coloring of $K_4$-minor-free graphs

A graph $G$ has a graph $H$ as a minor if $H$ can be obtained from a subgraph of $G$ by contracting edge, and $G$ is called $H$-minor-free if $G$ does not have $H$ as a minor.
Let $G$ be a normal $K_4$-minor-free graph with $\Delta \geq 4$, then $\chi'_a(G) \leq \Delta + 1$.

Let $G$ be a normal $K_4$-minor-free graph with $\Delta \geq 5$ and without adjacent $\Delta$-vertices, then $\chi'_a(G) = \Delta$. 
Theorem 6. (Wang and Wang, 2009) Let $G$ be a normal $K_4$-minor-free graph with $\Delta \geq 5$, then $\Delta \leq \chi'_a(G) \leq \Delta + 1$; and $\chi'_a(G) = \Delta + 1$ if and only if $G$ contains two adjacent $\Delta$-vertices.
IV Outline of Proofs

(4.1) Proof of Theorem 1

Lemma 1. Every outerplane graph $G$ with $|G| \geq 2$ contains one of (C1)-(C5) as follows:
(C1) A vertex $v$ of degree at most 3 is adjacent to a leaf.

(C2) A path $x_1x_2\cdots x_n$, $n \geq 4$, with $d_G(x_1) \neq 2$, $d_G(x_n) \neq 2$, and $d_G(x_i) = 2$ for all $i = 2, 3, \cdots, n-1$.

(C3) A $k$-vertex $v$, $k \geq 4$, is adjacent to a leaf and $k - 3$ vertices of degree $\leq 2$.

(C4) A 3-face $[uv_1v_2]$ satisfies $d_G(u) = 2$ and $d_G(v_1) = 3$.

(C5) Two 3-faces $[u_1v_1x]$ and $[u_2v_2x]$ satisfy $d_G(x) = 4$ and $d_G(u_1) = d_G(u_2) = 2$. 
Lemma 2. Every outerplane graph $G$ with $\Delta \leq 3$ contains one of (B1)-(B3):

(B1) A vertex $v$ adjacent to at most one vertex that is not a leaf.

(B2) A path $x_1x_2x_3x_4$ such that each of $x_2$ and $x_3$ is either a 2-vertex, or a 3-vertex that is adjacent to a leaf.

(B3) A 3-face $[uxy]$ such that either $d_G(u) = 2$, or $d_G(u) = 3$ and $u$ is adjacent to a leaf.
Lemma 3. Every outerplane graph $G$ with $\Delta = 4$ contains one of (A1)-(A4):

(A1) A vertex $v$ with $d_G(v) \neq 3$ is adjacent to a leaf.

(A2) A 3-vertex is adjacent to at least two leaves.

(A3) A path $x_1x_2x_3x_4$ such that each of $x_2$ and $x_3$ is either a 2-vertex, or a 3-vertex that is adjacent to a leaf.

(A4) A 3-face $[uxy]$ with $d_G(x) = 3$ such that either $d_G(u) = 2$, or $d_G(u) = 3$ and $u$ is adjacent to a leaf.
Lemma 4. Every outerplane graph $G$ with $\Delta = 3$ and without adjacent 3-vertices contains (D1)-(D2):

(D1) A leaf.

(D2) A cycle $C = x_1x_2\cdots x_n$, with $n \geq 3$, such that $d_G(x_1) = 3$ and $d_G(x_i) = 2$ for all $i = 2, 3, \cdots, n$. 
Theorem 1.1. If $G$ is an outerplane graph with $\Delta \leq 3$, then $\chi''_a(G) \leq 5$.

Proof. The proof proceeds by induction on $|T(G)| = |G| + ||G||$. If $|T(G)| \leq 5$, the theorem holds trivially. Suppose that $G$ is an outerplane graph with $\Delta \leq 3$ and $T(G) \geq 6$. By the induction assumption, any outerplane graph $H$ with $\Delta(H) \leq 3$ and $T(H) \leq T(G)$ has a total-5-AVD-coloring $f$. By Lemma 2, $G$ contains one of (B1)-(B3). We reduce each possible case to extend $f$ to the whole graph $G$. □
Theorem 1.2. If $G$ is an outerplane graph with $\Delta = 3$ and without adjacent 3-vertices, then $\chi''(G) = 4$.

Proof. By induction on $T(G)$. By Lemma 4, we handle possible case (D1) or (D2). □
Theorem 1.3. If $G$ is an outerplane graph with $\Delta \geq 4$, then $\chi''(G) \leq \Delta + 2$.

Proof. By induction on $T(G)$. By Lemma 1, we handle possible case (C1) or (C5). □
Theorem 1.4. If $G$ is an outerplane graph with $\Delta \geq 4$ and without adjacent $\Delta$-vertices, then $\chi''_a(G) = \Delta + 1$.

Proof. By induction on $T(G)$. By Lemma 3, we handle possible case (A1) or (A4). □
(4.2) Proof of Theorem 2

Theorem 2.1. If $G$ is a graph with $\text{mad}(G) < 3$ and $K(G) = \max\{\Delta + 2, 6\}$, then $\chi''_a(G) \leq K(G)$.

Proof. The proof proceeds by induction on $|T(G)| = |G| + ||G||$. If $|T(G)| \leq 5$, the theorem holds trivially. Suppose that $G$ is a graph with $\text{mad}(G) < 3$ and $T(G) \geq 6$. By the induction assumption, any proper subgraph $H$ of $G$ has a total-$K$-AVD-coloring $f$. □
Claim 1. No vertex of degree at most 3 is adjacent to a leaf.

Claim 2. There does not exist a path $P = x_1x_2 \cdots x_n$ with $d_G(x_1), d_G(x_n) \geq 3$ and $d_G(x_i) = 2$ for all $i = 2, 3, \cdots, n - 1$, where $n \geq 4$.

Claim 3. There does not exist a $k$-vertex $v$, $k \geq 4$, with neighbors $v_1, v_2, \cdots, v_k$ such that $d_G(v_1) = 1$, $d_G(v_i) \leq 2$ for $2 \leq i \leq k - 2$. 
Claim 4. There does not exist a 2-vertex $v$ adjacent to a 3-vertex $u$.

Claim 5. There does not exist a 4-vertex $v$ adjacent to three 2-vertices.

Claim 6. There does not exist a 5-vertex $v$ adjacent to five 2-vertices.
Let $H$ be the graph obtained by removing all leaves of $G$, then $\text{mad}(H) \leq \text{mad}(G) < 3$. $H$ has the following properties:
Claim 7.

(1) There are no vertices of degree less than 2;
(2) If \( v \in V(G) \) with \( 2 \leq d_G(v) \leq 3 \), then \( v \in V(H) \) and \( d_H(v) = d_G(v) \);
(3) If \( v \in V(H) \) with \( d_H(v) = 2 \), then \( d_G(v) = 2 \);
(4) If \( v \in V(G) \) with \( d_G(v) \geq 4 \), then \( d_H(v) \geq 3 \).

We make use of discharging method. First, we define an initial charge function \( w(v) = d_H(v) \) for all \( v \in V(H) \).
Next, we design a discharging rule and redistribute weights accordingly. Once the discharging is finished, a new charge function $w'$ is produced. However, the sum of all charges is kept fixed when the discharging is in progress. Nevertheless, we can show that $w'(v) \geq 3$ for all $v \in V(H)$. This leads to the following obvious contradiction:

$$3 = \frac{3|V(H)|}{|V(H)|} \leq \frac{\sum_{v \in V(H)} w'(v)}{|V(H)|} = \frac{\sum_{v \in V(H)} w(v)}{|V(H)|} = \frac{2|E(H)|}{|V(H)|} \leq \text{mad}(H) < 3.$$
The discharging rule is defined as follows:

(R) Every vertex $v$ of degree at least 4 gives $\frac{1}{2}$ to each adjacent 2-vertex.

Let $v \in V(H)$. Then $d_H(v) \geq 2$ by Claim 7(1).

If $d_H(v) = 2$, then $v$ is adjacent to two vertices of degree at least 4 by Claim 4, each of which sends $\frac{1}{2}$ to $v$ by (R). Thus, $w'(v) \geq d_H(v) + 2 \times \frac{1}{2} = 2 + 1 = 3$.

If $d_H(v) = 3$, then $w'(v) = w(v) = 3$. 
If $d_H(v) = 4$, then $v$ is adjacent to at most two 2-vertices by Claim 5. Thus, $w'(v) \geq 4 - 2 \times \frac{1}{2} = 3$.

If $d_H(v) = 5$, then $v$ is adjacent to at most four 2-vertices by Claim 6. Thus, $w'(v) \geq 5 - 4 \times \frac{1}{2} = 3$.

If $d_H(v) \geq 6$, then $v$ is adjacent to at most $d_H(v)$ 2-vertices and hence $w'(v) \geq d_H(v) - \frac{1}{2}d_H(v) = \frac{1}{2}d_H(v) \geq 3$ by (R). $\square$
Thank you for your attention!