Circular game chromatic number of graphs

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The **game chromatic number** $\chi_g(G)$ of $G$ is the least number of colours contained in $X$ so that Alice has a winning strategy for the colouring game on $G$ with the colour set $X$. 
The colouring game on planar graphs was first introduced by Steven Brams in 1981, and later the game was re-invented by Bodlaender in 1991, where the game chromatic number of an arbitrary graph is defined.
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The game chromatic number of various classes of graphs have been studied extensively in the literature.

For a class $\mathcal{K}$ of graphs, let $\chi_g(\mathcal{K}) = \max\{\chi_g(G) : G \in \mathcal{K}\}$. 
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Marking game and game colouring number

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For a vertex $x$ of $G$, let $b(x)$ be the number of neighbours of $x$ that are marked before $x$ is marked.

The score of the game is $1 + \max\{b(x) : x \in V(G)\}$. Alice’s goal is to minimize the score of the game, and Bob’s goal is to maximize the score.
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The *game colouring number* $\text{col}_g(G)$ of $G$ is the minimum $s$ such that Alice has a strategy that ensures that the resulting score is at most $s$.
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The game colouring number itself is also of independent interest, has application to graph packing, and has been studied extensively in the literature.
Circular chromatic number

For a positive real number $r$, let $S(r)$ denote the circle of circumference $r$. 

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Circular game chromatic number of graphs
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- For \( a, b \in S(r) \), 
  \([a, b]_r = \{x \in S(r) : 0 \leq [x - a]_r \leq [b - a]_r\}\),
  \((a, b)_r = \{x \in S(r) : 0 < [x - a]_r < [b - a]_r\}\). The length of the interval \([a, b]_r\) is equal to \([b - a]_r\).
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Two points $a, b \in S(r)$ partition $S(r)$ into two arcs: $[a, b]_r$ and $[b, a]_r$. The distance between $a$ and $b$, denoted by $|a - b|_r$, $\min\{[a - b]_r, [b - a]_r\} = \min\{|a - b|, r - |a - b|\}$. 
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- A circular $r$-colouring of $G$ is a mapping $f : V(G) \rightarrow S(r)$ such that for any edge $xy$ of $G$, $|f(x) - f(y)|_r \geq 1$. 
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- A circular $r$-colouring of $G$ is a mapping $f : V(G) \rightarrow S(r)$ such that for any edge $xy$ of $G$, $|f(x) - f(y)|_r \geq 1$.
- The circular chromatic number $\chi_c(G)$ of $G$ is the least $r$ for which $G$ has a circular $r$-colouring.
Circular colouring game

- Given a graph $G$ and a real number $r$, the *circular $r$-colouring game* on $G$ is a two-person game played by Alice and Bob. The two players alternate their turns.

The game ends if either all vertices of $G$ are coloured or there is an uncoloured vertex $x$ that has no legal colour. In the former case, Alice wins the game. In the latter case, Bob wins the game.

We need to specify who has the first move. We consider both Alice first version and Bob first version of circular colouring game.
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We need to specify who has the first move. We consider both Alice first version and Bob first version of circular colouring game.
For a graph $G$, let $R(G)$ be the set of real numbers $r$ for which Alice has a winning strategy in the circular $r$-colouring game on $G$. Therefore, $\chi_{cg}(G) \geq \chi_c(G)$. Equality holds for $K_1$ and for stars: for $n \geq 1$, $\chi_{cg}(K_1, n) = \chi_c(K_1, n) = 2$. 

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Observe that if $r \geq 2\Delta(G)$, it is obvious that Alice has a winning strategy for the circular $r$-colouring game on $G$. So $R(G) \neq \emptyset$, and hence $\chi_{cg}(G)$ is well-defined.
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If Alice wins the circular $r$-colouring game, then the players produce an $r$-circular colouring of $G$. Therefore

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Theorem 1
For any graph $G$, $\chi_{cg}(G) \leq 2\text{col}_g(G) - 2$. 

Theorem 2
For any graph $G$, $\chi_{Bcg}(G) \leq 2\text{col}_B(G) - 2$. 

Lemma 1
Let $G$ be a connected graph with at least one edge. If $G \neq K_1, n$ for any positive integer $n$, then $\chi_{cg}(G) \geq 4$. 

$\chi_{cg}(P_n) = 4$ for $n \geq 4$ and $\chi_{Bcg}(P_n) = 4$ for $n \geq 5$ and $\chi_{Bcg}(C_4) = 2$ and $\chi_{Bcg}(C_5) = 3$.
\( \chi_{cg}(G) \) and \( \text{col}_g(G) \)

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For any graph \( G \), \( \chi_{cg}(G) \leq 2\text{col}_g(G) - 2. \)

**Theorem 2**

For any graph \( G \), \( \chi^B_{cg}(G) \leq 2\text{col}_g^B(G) - 2. \)
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**Theorem 1**
For any graph \( G \), \( \chi_{cg}(G) \leq 2\text{col}_g(G) - 2 \).

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For any graph \( G \), \( \chi_{cg}^B(G) \leq 2\text{col}_g^B(G) - 2 \).

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- \( \chi_{cg}(P_n) = 4 \) for \( n \geq 4 \) and \( \chi_{cg}(C_n) = 4 \) for \( n \geq 3 \).
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Let $G$ be a connected graph with at least one edge. If $G \neq K_{1,n}$ for any positive integer $n$, then $\chi_{cg}(G) \geq 4$.

- $\chi_{cg}(P_n) = 4$ for $n \geq 4$ and $\chi_{cg}(C_n) = 4$ for $n \geq 3$.
- $\chi_{cg}^B(P_n) = 4$ for $n \geq 5$ and $\chi_{cg}^B(C_n) = 4$ for $n \geq 7$.
- $\chi_{cg}^B(P_4) = \chi_{cg}^B(C_4) = \chi_{cg}^B(C_6) = 2$ and $\chi_{cg}^B(C_5) = \chi_{cg}^B(C_3) = 3$. 
For a class $\mathcal{K}$ of graphs, let

$$\text{col}_g(\mathcal{K}) = \max\{\text{col}_g(G) : G \in \mathcal{K}\}.$$  
$$\chi_{cg}(\mathcal{K}) = \sup\{\chi_{cg}(G) : G \in \mathcal{K}\}.$$  

Recall that $\mathcal{P}$ is the class of planar graphs. Let $\mathcal{F}$ be the class of forests, $\mathcal{Q}$ be the class of outerplanar graphs and $\mathcal{P}\mathcal{K}_k$ be the class of partial $k$-trees.

It is known that $\text{col}_g(\mathcal{F}) = 4$, $\text{col}_g(\mathcal{Q}) = 7$, $\text{col}_g(\mathcal{P}) \leq 17$ and $\text{col}_g(\mathcal{P}\mathcal{K}_k) = 3k + 2$ for $k \geq 2$. 
Corollary 1

For the classes $\mathcal{F}, \mathcal{Q}, \mathcal{P}, \mathcal{PK}_k$ of graphs defined above,

$$\chi_{cg}(\mathcal{F}) \leq 6, \quad \chi_{cg}(\mathcal{Q}) \leq 12, \quad \chi_{cg}(\mathcal{P}) \leq 32, \quad \chi_{cg}(\mathcal{PK}_k) \leq 6k + 2.$$
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Theorem 3
For any positive real number $\varepsilon$, there is a tree $T$ with $\chi_{cg}(T) > 6 - \varepsilon$. Hence $\chi_{cg}(\mathcal{F}) = 6$. 

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A colouring of the vertices of a graph $G$ is *acyclic* if it is a proper colouring such that no cycle of $G$ is 2-coloured.

The minimum number of colours needed is the *acyclic chromatic number* of a graph $G$, denoted by $\chi_a(G)$.
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$$\chi_g(G) \leq \chi_a(G)(\chi_a(G) + 1).$$
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$$\chi_g(G) \leq \chi_a(G)(\chi_a(G) + 1).$$

**Theorem 4**

For any graph $G$, $\chi_{cg}(G) \leq \chi_a(G)(2\chi_a(G) + 2)$ and $\chi_{cg}^B(G) \leq \chi_a(G)(2\chi_a(G) + 2)$. 
The complete graphs

For a positive integer $n$, let

$$
\varphi(n) = \begin{cases} 
4k + 1, & \text{if } n = 3k + 1, \\
4k + 2, & \text{if } n = 3k + 2, \\
4k + 4, & \text{if } n = 3k + 3.
\end{cases}
$$

$$
\psi(n) = \begin{cases} 
4k, & \text{if } n = 3k + 1, \\
4k + 2, & \text{if } n = 3k + 2, \\
4k + 3, & \text{if } n = 3k + 3.
\end{cases}
$$

Theorem 5

$$
\chi_{cg}(K_n) = \varphi(n) \text{ and } \chi_{cg}^B(K_n) = \psi(n).
$$
Some open questions - 1

It is known that $\chi(G) = \lceil \chi_c(G) \rceil$. Is there any relation between $\chi_{cg}(G)$ and $\chi_g(G)$?

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Question 1

Are there functions $f, g : N \to R$ such that $g(\chi_g(G)) \leq \chi_{cg}(G) \leq f(\chi_g(G))$?

Question 2

Let $r$ and $r'$ be positive real numbers with $r < r'$. Suppose Alice has a winning strategy for the circular $r$-colouring game. Is it true that Alice also has a winning strategy for the circular $r'$-colouring game?
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Question 3

If $\chi_{cg}(G) = t$, is it true that Alice has a winning strategy for the circular $t$-colouring game on $G$?
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By Lemma 1, it is not difficult to see that no graph has circular game chromatic number in the open interval $(2, 4)$. 

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Circular game chromatic number of graphs
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Question 4
What are the possible value of $\chi_{cg}(G)$ for all graphs $G$?

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$$\chi_{cg}(\mathcal{F}) \leq 6, \ \chi_{cg}(Q) \leq 12, \ \chi_{cg}(P) \leq 32, \ \chi_{cg}(PK_k) \leq 6k + 2.$$  

It will be interesting to improve those bounds in Corollary 1 for these classes of graphs other than forests.
Thanks!
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