On \((d, 1)\)-Total Numbers of Graphs

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A joint work with
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All graphs mentioned in this talk are finite without self-loops or parallel edges.

Every graph is assumed to have at least one edge.

\[ f(x) - f(y) \geq \begin{cases} 
1 & \text{if vertices } x \text{ and } y \text{ are adjacent;} \\
1 & \text{if edges } x \text{ and } y \text{ are adjacent;} \\
d & \text{if vertex } x \text{ and edge } y \text{ are incident.} 
\end{cases} \]
(\(d, 1\))-total labeling

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**Definition**

A \((d, 1)\)-total labeling of a graph \(G = (V, E)\) is a function \(f\) from \(V(G) \cup E(G)\) to nonnegative integers such that

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1 & \text{if edges } x \text{ and } y \text{ are adjacent;} \\
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\end{cases}
\]
A \((d, 1)\)-total labeling taking values in the set \(\{0, 1, \ldots, k\}\) is called a \([k]-(d, 1)\)-total labeling.

The span of a \((d, 1)\)-total labeling is the maximum difference between two labels.

The minimum \(k\) among all \([k]-(d, 1)\)-total labellings of \(G\), denoted \(\lambda^T_d(G)\), is called the \((d, 1)\)-total number of \(G\).

\(\lambda^T_2(G)\) is the minimum span among all \(L(2, 1)\)-labelings of the subdivision graph \(G^S\) of a graph \(G\).
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Let $\Delta(G)$ denote the maximum degree of the graph $G$. Havet and Yu (2002) proposed the following conjecture.

\[(d, 1)\text{-Total Labeling Conjecture}\]

\[\lambda_d^T(G) \leq \min\{\Delta(G) + 2d - 1, 2\Delta(G) + d - 1\}.\]

Note that $\lambda_d^T(G) + 1$ is equal to the total chromatic number $\chi''(G)$ of the graph $G$.

When $d = 1$, the $(d, 1)$-total labeling conjecture is equivalent to the Total Coloring Conjecture proposed by Behzad (1965) and independently by Vizing (1968).
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Definition

Let $\chi(G)$, or $\chi'(G)$, denote the smallest number of colors needed to color the vertices, respectively the edges, of $G$ so that adjacent elements receive distinct colors.

Definition

- If each edge $e$ of $G$ is assigned a list $L(e)$ of possible colors and $G$ has a proper edge-coloring $\phi$ such that $\phi(e) \in L(e)$ for all $e \in E(G)$, then we say that $G$ is $L$-edge-colorable.

- The graph $G$ is said to be $k$-edge-choosable if it is $L$-edge-colorable for every assignment $L$ satisfying $|L(e)| = k$ for all $e \in E(G)$.

- Let $\chi'_l(G)$ denote the smallest $k$ such that $G$ is $k$-edge-choosable.
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An upper bound

The following lemma was proved by Havet and Yu (2008) and the case for $d = 2$ was proved by Wittlesey, Georges, and Mauro (1995).

**Lemma.**

\[ \lambda^T_d(G) \leq \chi(G) + \chi'(G) + d - 2. \]

\[ \lambda^T_d(G) \leq 2\Delta(G) + d - 1. \]

**Theorem.**

\[ \lambda^T_d(G) \leq \chi'_l(G) + 4d - 3. \]
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Theorem. (Borodin, Kostochka, and Woodall (1997))
\[ \chi'_l(G) \leq \left\lfloor \frac{3}{2} \Delta(G) \right\rfloor. \]

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An upper bound conjecture

Theorem.

Let $G$ be a graph with $\chi(G) = k$ and $\chi'(G) = k'$. If $k \geq 3d$, then

$$\lambda^T_d(G) \leq s + k' - 1,$$

where $s$ is equal to $4d - 2$ when $k = 3d$ or $3d + 1$, and equal to $\lceil (k + 9d - 5)/3 \rceil$ when $k \geq 3d + 2$.

This theorem implies that the following conjecture holds when $\chi(G) \geq 3d$.

Conjecture.

Let a graph $G$ satisfy $\chi(G) > \max\{2, d\}$. Then

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Let $G$ be a graph with $\chi(G) = k$ and $\chi'(G) = k'$. If $k \geq 3d$, then $\lambda_d^T(G) \leq s + k' - 1$, where $s$ is equal to $4d - 2$ when $k = 3d$ or $3d + 1$, and equal to $\lceil (k + 9d - 5)/3 \rceil$ when $k \geq 3d + 2$.

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The known values of $\lambda_d^T(K_n)$ computed by Havet and Yu supports the above conjecture.

**Corollary.**

Let $G$ be a bipartite graph. Then $\Delta(G) + d - 1 \leq \lambda_d^T(G) \leq \Delta(G) + d$ and $\lambda_d^T(G) = \Delta(G) + d$ when $d \geq \Delta(G)$ or $G$ is regular.

Since $\chi'(G) = \Delta(G)$ for a bipartite graph $G$, a consequence of this corollary is $\lambda_d^T(G) = \Delta(G) + d = \chi(G) + \chi'(G) + d - 2$ for a bipartite regular graph $G$. This together with $\lambda_4^T(K_4) = 9$ show that the assumption $\chi(G) > \max\{2, d\}$ cannot be removed.
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Lemma.

(1) $\lambda^T_d(G) \geq \Delta(G) + d - 1$.
(2) If $\lambda^T_d(G) = \Delta(G) + d - 1$, then each vertex of maximum degree is labeled with 0 or $\Delta(G) + d - 1$ in any $[\Delta(G) + d - 1]$-$(d, 1)$-total labeling.

Theorem

The following statements are equivalent.
(1) $m \geq \min\{2n, n + 2d - 1\}$ and $m \geq n + d$.
(2) There exists an $[m + d - 1]$-$(d, 1)$-total labeling $f$ for $K_{m,n}$ such that $f(x) = 0$ for all $x \in X$, or $f(x) = m + d - 1$ for all $x \in X$. 

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On $(d, 1)$-Total Numbers of Graphs
Complete bipartite graphs

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Complete bipartite graphs

Theorem

If $2n \leq m < n + d$, then $\lambda_d^T(K_{m,n}) = m + d$.

Theorem

Suppose that $m < \min\{2n, n + 2d - 1\}$ and $\lambda_d^T(K_{m,n}) = m + d - 1$. Then all the following statements hold.

1. $m \geq 3d + 1$.
2. $(n - m + 3d - 1)(2n - m) \leq nd$.
3. $m \geq n + d$.
4. $n/m \leq (\alpha + 1)/(\alpha + 2)$, where $\alpha = \lfloor (m - d - 2)/(2d - 1) \rfloor$.

Corollary

If $m < n + d$, then $\lambda_d^T(K_{m,n}) = m + d$. 
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If $m < n + d$, then $\lambda_d^T(K_{m,n}) = m + d$. 
Theorem

Let $1 \leq n \leq m$. Then

$$\lambda_1^T(K_{m,n}) = \chi''(K_{m,n}) - 1 = m + \delta_{m,n},$$

where $\delta_{m,n}$ denotes the Kronecker delta, i.e., its value is 1 if $m = n$ and is 0 otherwise.
Theorem

Let $1 \leq n \leq m$. Then

$$\lambda_2^T(K_{m,n}) = \begin{cases} 
  m + 2 & \text{if } m \leq n + 1, \text{ or } \\
  m + 1 & \text{otherwise.} 
\end{cases}$$
Theorem

Let $1 \leq n \leq m$. Then

$$\lambda_3^T(K_{m,n}) = \begin{cases} 
  m + 3 & \text{if } m \leq n + 2, \text{ or} \\
  m = n + 3 \text{ and } n \geq 4, \text{ or} \\
  m = n + 4 \text{ and } n = 5, 9, 10, 13, 14, 15; \\
  m + 2 & \text{otherwise.} 
\end{cases}$$
When \( n \) is one of the numbers 6, 7, 8, 11, 12, or 16, an \([n + 6]-(3, 1)\)-total labeling for \( K_{n+4,n} \) is given below by a table. The notation used is as follows. The label of the \( i \)-th row is assigned to the vertex \( x_i \in X \). The label of the \( j \)-th column is assigned to the vertex \( y_j \in Y \). The label at the \((i, j)\) cell is assigned to the edge \( x_iy_j \).

\[
\begin{array}{cccccccccc}
K_{10,6} & 6 & 6 & 9 & 9 & 11 & 11 & 1 & 1 & 1 & 1 \\
12 & 2 & 0 & 1 & 3 & 5 & 4 & 7 & 8 & 9 & 6 \\
12 & 3 & 1 & 0 & 2 & 6 & 5 & 9 & 4 & 8 & 7 \\
12 & 9 & 2 & 3 & 0 & 1 & 6 & 8 & 7 & 4 & 5 \\
0 & 10 & 3 & 4 & 6 & 8 & 7 & 12 & 11 & 5 & 9 \\
0 & 11 & 9 & 5 & 4 & 7 & 3 & 10 & 12 & 6 & 8 \\
0 & 12 & 10 & 6 & 5 & 3 & 8 & 11 & 9 & 7 & 4 \\
\end{array}
\]
### $K_{11,7}$, and $K_{12,8}$

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