Existence of Subdivisions of Vertex-Disjoint Graphs with Few Edges

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An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a chord of that cycle.
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A **unicyclic graph** is a connected graph which has exactly one cycle.
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If $H$ is a graph such that each of the components contains at most one cycle, then a cyclic subdivision of $H$ is defined as a subdivision of $H$ in which only edges on the cycle of $H$ are subdivided.
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A set of subgraphs of a graph $G$ is said to be vertex-disjoint if no two of them have a common vertex in $G$. 
Theorem (Dirac, 1952)

Let $G$ be a graph on $n \geq 3$ vertices. If $\delta(G) \geq n/2$, then $G$ contains a Hamilton cycle.
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This theorem can be restated as:

Theorem

Let \( G \) be a graph on \( n \geq 3 \) vertices. If \( \delta(G) \geq n/2 \) and \( C \) is a cycle of \( G \), then \( G \) contains a spanning subdivision of \( C \).
Theorem (Corrádi and Hajnal, 1963)

Let $s$ be a positive integer. If $G$ is a graph with at least $3s$ vertices such that $\delta(G) \geq 2s$, then $G$ contains $s$ vertex-disjoint cycles.
Theorem (Corrádi and Hajnal, 1963)

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This theorem can be restated as:

Theorem

Let $s$ be a positive integer and $H$ be the disjoint union of $s$ cycles of length 3. If $G$ is a graph with at least $3s$ vertices such that $\delta(G) \geq 2s$, then $G$ contains a subdivision of $H$. 
In two recent papers published in DM and G&C, Babu and Diwan gave generalizations of the theorems of Dirac and of Corrádi and Hajnal, in terms of subdivisions of graphs with few edges.
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In the following we will introduce three new results on the existence of subdivisions of vertex-disjoint graphs with few edges.
Theorem (Dirac, 1952)

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Babu and Diwan gave a generalization of this theorem by replacing the cycle $C$ with disjoint trees and unicyclic graphs.

Theorem (Babu and Diwan, DM, 2008)

Let $G$ be a graph on $n$ vertices and $H$ be a subgraph of $G$ such that each component of $H$ is either a non-trivial tree or a unicyclic graph. If the number of tree components of $H$ is $t$ and $\delta(G) \geq (n - t)/2$, then $G$ contains a spanning subdivision of $H$. 
Motivated by the above theorem of Babu and Diwan, we give a generalization of Dirac’s theorem in terms of subdivisions of vertex-disjoint cycles and chorded cycles.
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**Theorem (Qiao and Zhang, 2009)**

Let $G$ be a graph on $n \geq 3$ vertices and $H$ be a subgraph of $G$ such that each component of $H$ is a cycle with at most one chord. If $\delta(G) \geq n/2$, then $G$ contains a spanning subdivision of $H$ where only non-chord edges are subdivided.
Theorem (Corrádi and Hajnal, 1963)

Let $s$ be a positive integer and $H$ be the disjoint union of $s$ cycles of length 3. If $G$ is a graph with at least $3s$ vertices such that $\delta(G) \geq 2s$, then $G$ contains a subdivision of $H$. 
Theorem (Corrádi and Hajnal, 1963)

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Theorem (Brandt, 1994)

Let $n, r$ be two positive integers and $F$ be a forest with $n$ vertices and $r$ components. If $G$ is a graph with at least $n$ vertices such that $\delta(G) \geq n - r$, then $G$ contains a subgraph isomorphic to $F$. 
Theorem (Schuster, 1998)

Let $n, r, s$ be three positive integers, $F$ be a forest with $n$ vertices and $r$ components without isolated vertices, and $G$ be a graph with at least $n + 3s$ vertices such that $\delta(G) \geq (n - r) + 2s$. Then $G$ contains a vertex-disjoint union of a subgraph isomorphic to $F$ and $s$ cycles.
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Theorem (Babu and Diwan, G&C, 2008)

Let $n, r$ be two positive integers and $H$ be a graph with $n$ vertices and $r$ non-trivial components such that either a non-trivial tree or a unicyclic graph. If $G$ is a graph with at least $n$ vertices such that $\delta(G) \geq n - r$, then $G$ contains a cyclic subdivision of $H$. 
Theorem (Finkel, 2008)

Let $t$ be a positive integer. If $G$ is a graph with at least $4t$ vertices such that $\delta(G) \geq 3t$, then $G$ contains $t$ vertex-disjoint chorded cycles.
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Theorem (Qiao and Zhang, 2009)

Let $n, r, t$ be three non-negative integers. Let $H$ be a graph with $n$ vertices and $r$ components such that each component is either a non-trivial tree or a unicyclic graph. If $G$ is a graph with at least $n + 4t$ vertices such that $\delta(G) \geq (n - r) + 3t$, then $G$ contains a vertex-disjoint union of a cyclic subdivision of $H$ and $t$ chorded cycles.
Theorem (Wang, 2000)

Let $G$ be a bipartite graph with bipartition $(X, Y)$ such that $|X| = |Y| \geq sk$, where $s \geq 3$ and $k \geq 1$ are two integers. If $\delta(G) \geq (s - 1)k + 1$, then $G$ contains $k$ vertex-disjoint cycles of length at least $2s$. 
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Theorem (Babu and Diwan, G&C, 2008)

Let $H$ be a graph with $n$ vertices and $k$ non-trivial components such that either a non-trivial tree or a unicyclic graph. Let $G$ be a graph with at least $n$ vertices. If $\delta(G) \geq n - k$, then $G$ contains a cyclic subdivision of $H$. 
Theorem (Qiao and Zhang, 2009)

Let $H$ be a graph of order $n$ with $k$ components, each of which is an even cycle of length at least 6. Suppose that $G$ is a bipartite graph with bipartition $(X, Y)$ such that $|X| = |Y| \geq n/2$. If $\delta(G) \geq n/2 - k + 1$, then $G$ contains a subdivision of $H$. 
Theorem (Qiao and Zhang, 2009)

Let $H$ be a graph of order $n$ with $k$ components, each of which is an even cycle of length at least 6. Suppose that $G$ is a bipartite graph with bipartition $(X, Y)$ such that $|X| = |Y| \geq \frac{n}{2}$. If $\delta(G) \geq \frac{n}{2} - k + 1$, then $G$ contains a subdivision of $H$.

- This theorem is a generalization of Wang’s theorem when taking each cycle as a $C_6$. 
Conjecture (Wang, 1996)

Let $G$ be a balanced bipartite graph with bipartition $(X, Y)$ such that $|X| = |Y| = 2k$, where $k$ is a positive integer. If $\delta(G) \geq k + 1$, then $G$ contains $k$ vertex-disjoint cycles of length 4.
Conjecture (Wang, 1996)

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Conjecture

Let $H$ be a graph of order $n$ with $k$ components, each of which is an even cycle. Suppose that $G$ is a bipartite graph with bipartition $(X, Y)$ such that $|X| = |Y| \geq n/2$. If $\delta(G) \geq n/2 - k + 1$, then $G$ contains a subdivision of $H$. 
Thank you!