Coloring 1-planar graphs

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Outline

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What are 1-planar graphs?

Definition

A graph $G$ is **1-immersed** into a surface if it can be drawn on the surface so that each edge is crossed by at most one other edge. In particular, A graph is **1-planar** if it is 1-immersed into the plane (i.e. has a plane 1-immersion).
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What are 1-planar graphs?

**Note**

$K_7$, $K_{3,5}$ and $K_{4,4}$ are not 1-planar graphs.
What are 1-planar graphs?

Remark

The notion of 1-planar was introduced by Ringel in the connection with problem of the simultaneous coloring of adjacent/incidence of vertices and faces of plane graphs.

Definition

A vertex-face \( r \)-coloring is a mapping that assigns a color from the set \( \{1, \cdots, r\} \) to every vertex and every face of \( G \) such that different colors are assigned whenever two elements are either adjacent or incident.
What are 1-planar graphs?

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Example

The vertex-face chromatic number of $G$ is equal to the vertex chromatic number of $H$. 
Subclasses of 1-planar graphs

Definition
A graph $H$ is a **minor** of a graph $G$ if $H$ can be obtained from $G$ by contracting a subset of the edges in a subgraph of $G$.

Three subclasses of 1-planar graphs

1. Planar Graphs
   $\{K_5, K_{3,3}\}$-minor-free graphs
2. Series-Parallel Graphs
   $\{K_4\}$-minor-free graphs
3. Outerplanar Graphs
   $\{K_4, K_{2,3}\}$-minor-free graphs
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Can 1-planarity be characterized by forbidding some minors?

Observation [Korzhik, 2008]

There is a 1-planar subdivision of every graph $G$. Indeed, take a plane drawing of $G$ and then for every edge with at least two crossing points, place on the edge a new 2-valent vertex between every pair of adjacent crossing points. We obtain a 1-planar subdivision of $G$. 

![Diagram showing the transformation of a graph into a 1-planar subdivision](image)
Can 1-planarity be characterized by forbidding some minors?

Conclusion

The set of 1-planar graphs is not closed under taking minors.

\[ G \rightarrow \text{a 1-planar subdivision of } G \]
How many edges are there in a 1-planar graphs?

**Theorem A [Albertson and Mohar, 2006]**

For each graph 1-immersed on a surface with Euler characteristic $\varepsilon$, $e(G) \leq 4(v(G) - \varepsilon)$ holds. In particular, the number of edges of 1-planar graph $G$ is bounded by $4v(G) - 8$.

**Theorem B**

Let $G$ be a graph 1-immersed on a surface with Euler characteristic $\varepsilon$. We have $e(G) \leq \frac{2g(G)-2}{g(G)-2} (v(G) - \varepsilon)$. 

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**Corollary**

1. Each 1-planar graph has a vertex of degree $\leq 7$. Then bound 7 is best possible.

2. For each triangle-free 1-planar graph $G$, $e(G) \leq 3v(G) - 6$ holds.

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**Definition**

We say that $G$ is *$f$-choosable* if, whenever we are given lists $A_v$ of colors with $|A_v| = f(v)$ for each $v \in V(G)$, we can choose a color $c(v) \in A_v$ for each vertex $v$ such that no two adjacent vertices receive the same color. The *list chromatic index* $\chi'_{list}(G)$ of $G$ is the smallest number such that $G$ is *$f$-choosable* when we assign $f(x) = k$ for each $v \in V(G)$.

**Remark**

We can similar define the list analogue of some other colorings, such as *list edge coloring*, *list total coloring*, etc.
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We can similar define the list analogue of some other colorings, such as list edge coloring, list total coloring, etc.
Theorem C [Borodin, 1984]
Each 1-planar graph is 6-colorable.

Theorem D [Ringel, 1981]
If $G$ can be 1-immersed into an orientable surface $S_g$ of genus $g \geq 1$, then $\chi(G) \leq \lceil (9 + \sqrt{64g + 17})/2 \rceil$. 
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Theorem E [Albertson and Mohar, 2006]

If $G$ can be 1-immersed into a surface of Euler genus $g$, then $\chi'_{\text{list}}(G) \leq R(g) = \lfloor (9 + \sqrt{32g + 17})/2 \rfloor$. Moreover, if $g = 2$ or $g \geq 4$, then $\chi'_{\text{list}}(G) = R(g)$ if and only if $G$ contains the complete graph of order $R(g)$ as a subgraph.

Theorem F [Borodin, Kostochka, Raspaud and Sopena, 2001]

Each 1-planar graph is (list) acyclically 20-colorable.
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Theorem F [Borodin, Kostochka, Raspaud and Sopena, 2001]

Each 1-planar graph is (list) acyclically 20-colorable.
Definition

A proper coloring of a graph $G$ is called **acyclic** if there is no bicolored cycle in $G$.

Theorem F [Borodin, Kostochka, Raspaud and Sopena, 2001]

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Our Results: Edge Coloring of 1-PG

Vizing’s Theorem on Edge Coloring

For any graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Definition

A graph $G$ is said to be of class one if $\chi'(G) = \Delta(G)$, and of class two if $\chi'(G) = \Delta(G) + 1$. 
Our Results: Edge Coloring of 1-PG

1. A graph $G$ which can be 1-immersed on a surface with Euler characteristic $\varepsilon$ is of class one provided one of the following conditions holds:
   (a) $\varepsilon \geq 0$ and $\Delta(G) \geq 11$;
   (b) $\varepsilon < 0$ and $\Delta(G) > \frac{19 + \sqrt{169 - 144\varepsilon}}{3}$.

2. Let $G$ be a triangle-free graph which can be 1-immersed on a plane or a projective plane. Then $G$ is of class one if $\Delta(G) \geq 8$. 

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Our Results: Edge Coloring of 1-PG

Edge coloring of 1-immersed graphs

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Our Results: List Edge and List Total Coloring of 1-PG

Total coloring conjecture (TCC)

For any graph $G$, $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$.

List edge/total coloring conjecture (LECC/LTCC)

For any graph $G$,

(a) $\chi'_{list}(G) = \chi'(G)$ and

(b) $\chi''_{list}(G) = \chi''(G)$. 
Our Results: List Edge and List Total Coloring of 1-PG

List edge coloring and list total coloring of 1-planar graphs

1. Let $G$ be a 1-planar graph with maximum degree $\Delta \geq 16$. Then $\chi'_{list}(G) \leq \Delta + 1$ and $\chi''_{list}(G) \leq \Delta + 2$.

2. Let $G$ be a 1-planar graph with maximum degree $\Delta \geq 21$. Then $\chi'_{list}(G) = \Delta$ and $\chi''_{list}(G) = \Delta + 1$. 
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List edge coloring and list total coloring of 1-planar graphs

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2. Let $G$ be a 1-planar graph with maximum degree $\Delta \geq 21$. Then $\chi'_{list}(G) = \Delta$ and $\chi''_{list}(G) = \Delta + 1$.
Conclusion

1. TCC holds for 1-planar graph with maximum degree $\Delta(G) \geq 16$.

2. LECC and LTCC holds for 1-planar graph with maximum degree $\Delta(G) \geq 21$. 
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Coloring 1-planar graphs
Our Results: Acyclic Edge Coloring of 1-PG

Definition

A proper $k$-edge coloring $c$ of $G$ is called an **acyclic $k$-edge-coloring** of $G$ if there are no bichromatic cycles in $G$ under the coloring $c$. The smallest number of colors such that $G$ has an acyclic edge coloring is called the **acyclic chromatic number** of $G$, denoted by $\chi'_a(G)$. We can also define the list analogue of acyclic $k$-edge-coloring similarly. The **list acyclic chromatic number** of $G$ is denoted by $\chi'_{la}(G)$. 
Our Results: Acyclic Edge Coloring of 1-PG

History

In 1991, Alon et al. proved that $\chi'_a(G) \leq 64\Delta(G)$ for any graph $G$ of maximum degree $\Delta(G)$. This bound was later improved to $16\Delta(G)$ by Molloy and Reed.
Our Results: Acyclic Edge Coloring of 1-PG

Acyclic edge coloring of planar graphs [Hou et al., 2009]

Let $G$ be a planar graph. Then
$$\chi'_a(G) \leq \max\{2\Delta(G) - 2, \Delta(G) + 22\}.$$ 

Problem

Let $G$ be a 1-planar graph and $C$ be the minimum integer such that $\chi'_a(G) \leq \max\{2\Delta(G) - 2, \Delta(G) + C\}$. Does such a constant $C$ exist?
Our Results: Acyclic Edge Coloring of 1-PG

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List acyclic edge coloring of 4-fold graphs

If $G$ is a graph such that $e(G') \leq 4\nu(G') - 1$ for each $G' \subseteq G$, then $\chi'_{la}(G) \leq 3\Delta + 70$.

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List acyclic edge coloring of 1-planar graphs

Let \( G \) be a 1-planar graph. Then \( \chi'_{la}(G) \leq 3\Delta + 70 \).
For Further Reading I


For Further Reading II


For Further Reading III

I. Fabrici and T. Madaras.  
The structure of 1-planar graphs.  

J. Hou, J. L. Wu, G. Liu and B. Liu  
Acyclic edge colorings of planar graphs and series-parallel graphs.  

Vladimir P. Korzhik.  
Minimal non-1-planar graphs.  
For Further Reading IV

G. Ringel  
A nine color theorem for the torus and klein bottle.  

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Submitted.

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Thanks for your attention!