Enumerating Typical Coverings of a Circulant Graph

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Enumerating Typical Coverings of a Circulant Graph

Rongquan Feng
Throughout this talk, graphs are finite, undirected, simple (may have loops somewhere) and connected. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, $u$ and $v$ are said to be adjacent, denoted by $u \sim v$, if $uv \in E(G)$. The neighborhood of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to $v$. The valency (or degree) of the vertex $v$ is the cardinality of the set $N(v)$. An automorphism $\sigma$ of $G$ is a permutation of the vertex set $V(G)$ that preserves adjacency. That is, $\sigma$ is a bijection of $V(G)$ to itself such that $u \sim v$ if and only if $\sigma(u) \sim \sigma(v)$. The set of automorphisms forms a permutation group under the composition of maps, called the...
Graphs II

*full automorphism group* of the graph $G$ which is denoted by $\text{Aut}(G)$. Any subgroup of $\text{Aut}(G)$ is called an automorphism group of $G$. A graph $G$ is *vertex-transitive* if $\text{Aut}(G)$ acts transitively on the vertex set $V(G)$, *i.e.*, for any $u, v \in V(G)$, there exists an automorphism of $G$ which maps $u$ to $v$.

Let $G_1$ and $G_2$ be two graphs. A *homomorphism* $\sigma$ is a surjection $\sigma : V(G_1) \to V(G_2)$ such that if $u$ and $v$ are adjacent vertices in $G_1$ then $\sigma(u)$ and $\sigma(v)$ are adjacent vertices in $G_2$. An *isomorphism* is a bijection $\sigma$ such that both $\sigma$ and $\sigma^{-1}$ are homomorphisms.
Cayley graphs

Let \( A \) be a finite group and let \( X \) be a subset of \( A \) such that \( X = X^{-1} \) (called symmetric) and \( 1 \notin X \). The **Cayley graph** \( G = \text{Cay}(A, X) \) on \( A \) relative to \( X \) is the graph having vertex set \( V(G) = A \) and two vertices \( a, b \in A \) are adjacent iff \( b = ax \) for some \( x \in X \). Sometimes we write \( x_a \) for the edge \( \{a, ax\} \). The elements of \( X \) are called the *connectors* of the Cayley graph \( \text{Cay}(A, X) \) and the set \( X \) is called a *connecting set*. 
Circulant graphs

A circulant graph is a Cayley graph on a cyclic group $\mathbb{Z}_n$. Circulant graphs are widely applied to telecommunication networks, VLSI design and distributed computation. They are usually used as topologies and are called loop networks or chordal rings.

A circulant graph $\text{Cay}(\mathbb{Z}_n, X)$ is of odd valency if and only if $n$ is even and the order 2 element $\frac{n}{2} \in \mathbb{Z}_n$ is a connector, and that if a circulant graph $\text{Cay}(\mathbb{Z}_n, X)$ with $X = \{\pm i_1, \pm i_2, \ldots, \pm i_k\}$ is connected, then $(i_1, i_2, \ldots, i_k, n) = 1$, where $(i_1, i_2, \ldots, i_k, n)$ denotes the greatest common divisor of $i_1, i_2, \ldots, i_k$ and $n$. Also, we identify the integers $0, 1, \ldots, n - 1$ with their residue classes modulo $n$. 
Regular quotients and regular coverings I

Let $G$ be a graph and let $\mathcal{B}$ be a group. An action of the group $\mathcal{B}$ on the graph $G$ is a group homomorphism from $\mathcal{B}$ to $\text{Aut}(G)$. Let $\phi$ be such an action. Denote by $\phi_b$ the image of $b \in \mathcal{B}$ under $\phi$. This is equivalent to that for each element $b \in \mathcal{B}$, there is an automorphism $\phi_b \in \text{Aut}(G)$ with the following two conditions:

1. if 1 is the group identity in $\mathcal{B}$, then $\phi_1$ is the identity automorphism, and

2. for all $b, c \in \mathcal{B}$, we have $\phi_b \circ \phi_c = \phi_{bc}$.

Moreover, if for every $b \in \mathcal{B}$, $b \neq 1$, there is no vertex $v$ such that $\phi_b(v) = v$ and no edge $e$ such that $\phi_b(e) = e$, then the group $\mathcal{B}$ is said to act freely on $G$. 
For any vertex \( v \), the orbit \([v]\) is defined as

\[ [v] = \{ \phi_b(v) \mid b \in \mathcal{B} \}. \]

Similarly, for any edge \( e \), define

\[ [e] = \{ \phi_b(e) \mid b \in \mathcal{B} \}. \]

The sets of vertex orbits and edge orbits are denoted \( V/\mathcal{B} \) and \( E/\mathcal{B} \), respectively. It is clear that the vertex orbits partition the vertex set \( V(G) \) and the edge orbit partition the edge set \( E(G) \). Also it is clear that if \( \mathcal{B} \) act freely on \( G \), then the size of each orbit (vertex or edge) is just the order of the group \( \mathcal{B} \).
Regular quotients and regular coverings III

Example 1

Let $G$ be the Petersen graph and let $\mathcal{B} = \mathbb{Z}_5$. For any $i \in \mathbb{Z}_5$, let $\phi_i$ be the rotation of $2\pi i/5$ radians. Then $\mathbb{Z}_5$ acts freely on $G$. Under this action, there are two vertex orbits. One containing the outer 5 vertices and the other containing the inner 5 vertices. There are 3 edge orbits. One containing the 5 edges of the outer pentagon, one containing the 5 edges of the inner star, and one containing the 5 edges that run between the star and the pentagon.

Figure 1: Petersen graph
The regular quotient $G/B$ is defined to be the graph with vertex set $V/B$ and two orbits $[u]$ and $[v]$ are adjacent in $G/B$ if and only if there are $u_1 \in [u]$ and $v_1 \in [v]$ such that $u_1 \sim v_1$ in $G$.

**Example 2**

The regular quotient of the Petersen graph under the prescribed action of $\mathbb{Z}_5$ is the dumbbell graph.

Figure 2: the dumbbell graph
Regular quotients and regular coverings

Associated with a graph quotient there is a vertex function \( v \mapsto [v] \) and an edge function \( e \mapsto [e] \) that together are called the *quotient map* \( q_B : G \to G/B \).

A graph map \( p : \tilde{G} \to G \) is called a *regular covering projection* (or *regular covering*) if it is isomorphic to a quotient map in the following sense: there exists a group \( B \) which acts freely on \( \tilde{G} \), and there is a graph isomorphism \( i : \tilde{G}/B \to G \) such that \( i \circ q_B = p \), where \( q_B : \tilde{G} \to \tilde{G}/B \) is the quotient map. In this case, the graph \( \tilde{G} \) is also called a *regular covering* of the graph \( G \) and the graph \( G \) is called the base graph.

\[
\begin{array}{ccc}
\tilde{G} & \xrightarrow{q_B} & \tilde{G}/B \\
\downarrow{p} & \circlearrowleft & \downarrow{i} \\
G & & G
\end{array}
\]
Example 3

The 3-cube $Q_3 = \text{Cay}(A, X)$ with $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $X = \{(1,0,0), (0,1,0), (0,0,1)\}$. For any $i \in \mathbb{Z}_2$, define $\phi_i : (a_1, a_2, a_3) \mapsto (a_1 + i, a_2 + i, a_3 + i)$. Then $\mathbb{Z}_2$ acts freely on $Q_3$. The quotient is the complete graph $K_4$. Thus $Q_3$ is a regular covering of $K_4$. 
Ordinary voltages I

Let $G$ be a graph. Every edge of a graph $G$ gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse directed edge to a directed edge $e = uv$. A directed edge is also called an arc and the set of arcs of the graph $G$ is denoted by $D(G)$.

Let $\mathcal{A}$ be a finite group. An ordinary voltage assignment $\alpha$ of $G$ is a function $\alpha : D(G) \to \mathcal{A}$ with the property that $\alpha(e^{-1}) = \alpha(e)^{-1}$ for each $e \in D(G)$. The pair $\langle G, \alpha \rangle$ is called an ordinary voltage graph. The values of $\alpha$ are called voltages and $\mathcal{A}$ is called the voltage group.
Ordinary voltages II

The derived graph $G^\alpha$ from a voltage assignment $\alpha$ is defined as follows: $V(G^\alpha) = V(G) \times A$, and two vertices $(u, a)$ and $(v, b)$ are adjacent in $G^\alpha$ if and only if $u$ and $v$ are adjacent in $G$ and $b = a\alpha(uv)$. Note that $(u, a) \sim (v, b)$ iff $(v, b) \sim (u, a)$. So the derived graph $G^\alpha$ is undirected.

$$G^\alpha : (u, a) \rightarrow (v, b) \quad b = a\alpha(uv)$$

Example 4

Let $G$ be the dumbbell graph with voltages in the cyclic group $\mathbb{Z}_5$ described below. The derived graph is the Petersen graph.

$$G : \begin{array}{c}
\alpha(uv) \in A \\
\begin{array}{c}
u \\
v
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
2 \\
0 \\
1
\end{array}
\end{array}$$
Ordinary voltages III

Example 5

Let $G$ be the bouquet $B_n$ with $n$ circles and let $A$ be a finite group. Given voltages to those $n$ circles as $x_1, x_2, \ldots, x_n$. Then the derived graph $G^\alpha$ is just the Cayley graph on $A$ relative to $X = \{x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$. Be careful for those $x_i$ with order 2.

Example 6

Let $\mathbb{Z}_2 = \{0, 1\}$ be the group of order 2. Define $\alpha : D(G) \rightarrow \mathbb{Z}_2$ as $\alpha(uv) = 1$ for any arc $uv$ in the graph $G$. The derived graph $G^\alpha$ is called the canonical double covering of $G$. It is easy to check that the canonical double covering of $K_4$ is the 3-cube $Q_3$. 
Ordinary voltages IV

If voltages are assigned in the group $\mathcal{A}$ to the base graph $G$, then for any $v \in V(G)$, the set of vertices $v_a = (v, a)$ for $a \in \mathcal{A}$ in the derived graph $G^\alpha$ is called the fiber over $v$. Also for any edge $e = uv \in E(G)$, the set of edges $e_a = (u_a, v_{a\alpha}(uv))$ for $a \in \mathcal{A}$ is called the fiber over $e$.

The first coordinate projection $p_\alpha : G^\alpha \to G$ is called the natural projection. That is $p_\alpha$ maps every vertex in the fiber over $v$ to the vertex $v$ for all $v \in V(G)$ and maps each edge in the fiber over $e$ to the edge $e$ for all $e \in E(G)$. 
Theorem 1

The natural projection of the derived graph onto the base graph is a regular covering projection.

Theorem 2

Let $A$ be a group acting freely on the graph $\tilde{G}$ and let $G$ be the resulting quotient graph. Then there is an assignment $\alpha$ of voltages in $A$ to the quotient graph $G$ and a labeling of the vertices of $\tilde{G}$ by the elements of $V(G) \times A$ such that $\tilde{G} = G^\alpha$ and that the given action of $A$ on $\tilde{G}$ is the natural left action of $A$ on $G^\alpha$. 
Graph coverings I

A *covering projection* (or simply *covering*) from a graph \( \tilde{G} \) to another \( G \) is a homomorphism \( p : \tilde{G} \to G \) such that \( p \) is a local isomorphism, i.e., \( p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v) \) is a bijection for all vertices \( v \in V(G) \) and \( \tilde{v} \in p^{-1}(v) \). The fibre of an edge or a vertex is its preimage under \( p \). That is, for any vertex \( v \in V(G) \) and any \( \tilde{v} \in p^{-1}(v) \), \( p \) maps the set of edges incident with \( \tilde{v} \) bijectively to the set of edges incident with \( v \). It is clear that the natural projection \( p_\alpha \) of an ordinary derived graph onto its base graph is a covering projection.
Graph coverings II

Let $p : \tilde{G} \to G$ be a covering and let $u$ and $v$ be adjacent vertices of $G$. Since $G$ does not have multiple edges, each vertex in $p^{-1}(u)$ must be joined to exactly one vertex in $p^{-1}(v)$, and vice versa. This shows that $p^{-1}(u)$ and $p^{-1}(v)$ have the same cardinality, $r$ say, and that the subgraph of $\tilde{G}$ induced by the vertices $p^{-1}(u) \cup p^{-1}(v)$ is an $r$-matching. Thus each vertex fibre has size $r$ since $G$ is connected. Sometimes, a graph $\tilde{G}$ is also called a covering of $G$ with the projection $p : \tilde{G} \to G$ and the graph $G$ is the base graph, and it is $r$-fold if $p$ is $r$-to-one, and $r$ is called the index (or folding number) of the covering.
Let $p : \tilde{G} \to G$ be an $r$-fold covering. It is clear that every fibre of a vertex is an $r$-independent set and every fibre of an edge is an $r$-matching in the graph $\tilde{G}$.

Example 7

The cube $Q_3$ is a double (2-fold) coverings of the complete graph $K_4$. Every pair of vertices at distance 3 of $Q_3$ maps to a vertex of $K_4$. 
Remark:

1. The concept of covering introduced here is the same as that which arises in Topology. That is, if we view a graph as real points joined by real lines, then our coverings would be called coverings by topologists. But an interesting observation is that a graph covering is similar as a field extension while a regular covering is similar to a normal field extension.

2. Godsil once said “Covers are surprisingly useful and interesting;...... Unfortunately there is no extensive treatment of them from a combinatorial view in the literature,....." (C.D. Godsil, Algebraic Combinatorics, Chapman & Hall, New York, 1993, pp 91–92.)
Coverings are closely related to antipodal distance-regular graphs.

Why regular coverings are called regular? The reason is that the group $B$ acts regularly (transitive with no fixed point) on each vertex fiber. In the regular covering case, each edge fibre (an $r$-matching) can be determined by one edge of the matching.

It is clear that every double covering is regular.

A regular covering of a regular covering of $G$ is not necessary a regular covering of $G$. Such an example can be find in “J.L. Gross and T.W. Tucker, Generating all graph coverings by permutation voltage assignments, Discrete Math. 18 (1977), 273–283.”
Graph coverings VI

Again every edge of a graph $G$ gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse directed edge to a directed edge $e = uv$. A directed edge is also called an *arc* and the set of arcs of the graph $G$ is denoted by $D(G)$.

Let $S_r$ be the symmetric group of degree $r$. A *permutation voltage assignment* $\alpha$ of $G$ is a function $\alpha : D(G) \rightarrow S_r$ with the property that $\alpha(e^{-1}) = \alpha(e)^{-1}$ for each $e \in D(G)$. 
Graph coverings VII

Let $\alpha$ be a permutation voltage assignment. The derived graph $G^\alpha$ from $\alpha$ is defined as follows:
$V(G^\alpha) = V(G) \times \{1, 2, \ldots, r\}$, and two vertices $(u, i)$ and $(v, j)$ are adjacent in $G^\alpha$ if $u$ and $v$ are adjacent in $G$ and $j = \alpha(uv)(i)$. The first coordinate projection $p_\phi: G^\phi \to G$ is a covering.

\[
G^\alpha: \quad (u, i) \rightarrow (v, j) \quad j = \alpha(uv)(i)
\]

$G: \quad \alpha(uv) \in S_r \quad u \rightarrow v$
The natural projection $p_\alpha : G^\alpha \to G$ for permutation voltage graph $\langle G, \alpha \rangle_r$ is the graph map that takes any vertex $v_i$ or edge $e_i$ of the derived graph to the vertex $v$ or edge $e$ of the base graph. The set of vertices $v_i$ for $i = 1, \ldots, r$ is called the fiber over $v$. Similarly, we have the fiber over an edge $e$. It is a straightforward exercise that the natural projection $p_\alpha : G^\alpha \to G$ associated with any permutation voltage graph $\langle G, \alpha \rangle_r$ is a covering projection.
Graph coverings IX

Theorem 3

Let the graph map \( q : \tilde{G} \rightarrow G \) be a covering projection. Then there is an assignment \( \alpha \) of permutation voltages to the base graph \( G \) such that the derived graph \( G^\alpha \) is isomorphic to \( G \).

Let \( C^1(G; r) \) denote the set of all permutation voltage assignments \( \alpha : D(G) \rightarrow S_r \) of \( G \). Then every \( r \)-fold covering \( \tilde{G} \) of a graph \( G \) can be derived from a permutation voltage assignment in \( C^1(G; r) \).

The details on graph coverings can be found in the book “J.L. Gross and T.W. Tucker, Topological Graph Theory, Wiley, New York, 1987”. (Or in Godsil’s book “Algebraic Combinatorics”.)
Graph coverings X

It is natural to ask which (regular) coverings of a graph $G$ have a property $\mathcal{P}$ when the base graph $G$ has this property $\mathcal{P}$. For example, every covering of a bipartite graph is bipartite (very easy!). But for other cases, the problem might be very difficult. Godsil and Hensel (JCTB 56 (1992), 205–238) considered $\mathcal{P}$ as being distance-regularity. They restricted the base graph $G$ to a complete graph. In this talk, we take for $\mathcal{P}$ the property of being circulant.
Isomorphism of coverings

Two coverings $p_i : \tilde{G}_i \to G$, $i = 1, 2$, are said to be isomorphic if there exists a graph isomorphism $\Phi : \tilde{G}_1 \to \tilde{G}_2$ such that $p_2 \circ \Phi = p_1$. Such a $\Phi$ is called a covering isomorphism.

\[
\begin{array}{c}
\tilde{G}_1 \\
\Phi \\
\tilde{G}_2
\end{array}
\quad
\begin{array}{c}
p_1 \\
\downarrow \\
G \\
\downarrow \\
p_2
\end{array}
\]

**Note:** Let $T$ be a spanning tree of the graph $G$. A permutation voltage assignment $\phi$ is said to be normalized with respect to the spanning tree $T$ if $\phi(uv)$ is identity for all arcs contained in $T$. Any covering of $G$ is isomorphic to a covering derived from a normalized permutation voltage assignment.
Trivalent circulant graphs

Theorem 4

No trivalent circulant graph can be a double covering of a simple graph. In particular, no trivalent circulant graph has any circulant double covering.

Proof: Let $\tilde{G}$ be a double covering of the graph $G$. For any vertex $a$ of $\tilde{G}$, let $\bar{a}$ denote the other vertex of $\tilde{G}$ which is in the same fibre as vertex $a$, and let $\overline{U} = \{\overline{x} \mid x \in U\}$ for a vertex set $U$. Then $\overline{a} = a$, $N(\overline{a}) = N(a)$, and the distance between $\bar{a}$ and $a$ is greater than 2. Let $\tilde{G} = \text{Cay}(\mathbb{Z}_{2n}, \{\pm i, n\})$. Then for any $a \in V(\tilde{G})$, one can prove that $\overline{a + n} = \overline{\bar{a} + n}$, $\overline{a + i} = \overline{\bar{a} + i}$ and $\overline{a - i} = \overline{\bar{a} - i}$, from which the result follows.
Folding Number \( \geq 3 \) I

Let \( G = \text{Cay}(\mathbb{Z}_n, \{\pm i, \frac{n}{2}\}) \) be a trivalent graph. If \( G \neq K_4 \) or \( K_{3,3} \) then any edge of \( G \) determined by the connector \( i \) is contained in a unique 4-cycle while an edge determined by the connector \( \frac{n}{2} \) is contained in exactly two 4-cycles.

Let \( p : \text{Cay}(\mathbb{Z}_{rn}, \{\pm j, \frac{rn}{2}\}) \rightarrow \text{Cay}(\mathbb{Z}_n, \{\pm i, \frac{n}{2}\}) \) be a circulant connected \( r \)-fold covering. Then \( p \) maps a 4-cycle in the covering graph to a 4-cycle in \( G \). Then we can prove the following theorem.
Theorem 5

Let $G$ be a trivalent circulant graph with $n$ vertices but $G \neq K_4$ or $K_{3,3}$. Then every circulant connected $r$-fold covering projection of $G$ is a group epimorphism from $\mathbb{Z}_{rn}$ onto $\mathbb{Z}_n$.

Remark: Such a covering in above Theorem is called typical.
Typical coverings of a Cayley graph I

A covering \( p : \text{Cay}(\mathcal{A}, X) \to \text{Cay}(\mathcal{Q}, Y) \) is *typical* if the projection \( p : \mathcal{A} \to \mathcal{Q} \) on the vertex sets is a group epimorphism. Thus \( p(1_\mathcal{A}) = 1_\mathcal{Q} \) and then \( p(X) = Y \). Conversely, let \( f : \mathcal{A} \to \mathcal{Q} \) be an epimorphism with kernel \( \mathcal{K} \) and \( Y = f(X) \), denote by \( f_* : \text{Cay}(\mathcal{A}, X) \to \text{Cay}(\mathcal{Q}, Y) \) the typical covering derived from \( f \). It is easy to see that the two graph coverings \( q_\mathcal{K} : \text{Cay}(\mathcal{A}, X) \to \text{Cay}(\mathcal{A}, X)/\mathcal{K} \) and
\[ f_* : \text{Cay}(\mathcal{A}, X) \to \text{Cay}(\mathcal{Q}, Y) \]
can be identified through a graph isomorphism \( f_# : \text{Cay}(\mathcal{A}, X)/\mathcal{K} \to \text{Cay}(\mathcal{Q}, Y) \) defined by \( a\mathcal{K} \mapsto f_*(a) \). In other words, \( f_* = f_# \circ q_\mathcal{K} \). So \( f_* \) is a regular covering.
Typical coverings of a Cayley graph II

**Note:** In a typical covering $f_*$ derived from $f$, if there is an $x \in X \cap K$, or there exist two distinct elements $x$ and $x'$ in $X$ such that $f(x) = f(x')$, then the graph $\text{Cay}(Q, Y)$ cannot be simple. Therefore, we assume that $X \cap K = \emptyset$ and $f(x) \neq f(x')$ for any $x \neq x'$ in $X$ in order to deal with only simple graphs throughout this talk.

**Goal:** Given a Cayley graph $G = \text{Cay}(Q, Y)$, enumerate the isomorphism classes of typical coverings of the graph $G$. It is clear that if the folding number is $r$ and the typical covering graph is $\text{Cay}(A, X)$, then the group $A$ must be an extension of a group of order $r$ by the group $Q$. 

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*Enumerating Typical Coverings of a Circulant Graph*

Rongquan Feng
Theorem 6

Let \( f : \text{Cay}(A_1, X_1) \rightarrow \text{Cay}(Q, Y) \) and \( g : \text{Cay}(A_2, X_2) \rightarrow \text{Cay}(Q, Y) \) be two connected typical coverings of the same graph \( \text{Cay}(Q, Y) \). Then, two typical coverings \( f \) and \( g \) are isomorphic if and only if there is a group isomorphism \( \Phi : A_1 \rightarrow A_2 \) such that \( g \circ \Phi = f \) and \( \Phi(X_1) = X_2 \).
Let us enumerate the isomorphism classes of typical circulant $r$-fold coverings of a circulant graph at first. That is, $\mathcal{Q} = \mathbb{Z}_n$ and $\mathcal{A} = \mathbb{Z}_{rn}$.

**Case I**: $r = p$ is a prime.

**Case I(1)**: $p = 2$.

If $\text{Cay}(\mathbb{Z}_{2n}, X)$ is a typical covering of the graph $\text{Cay}(\mathbb{Z}_n, Y)$, then $|X|$ must be even because if it is odd, then $n \in X \cap \mathcal{K}$. The enumeration result is listed below.
Theorem 7 (F, Kwak, 2004)

Let $G$ be a connected circulant graph of order $n$. If the valency of $G$ is odd, there is no typical circulant double covering of $G$. If the valency of $G$ is even, say $2k$, then, the number of isomorphism classes of typical circulant connected double coverings of $G$ is $2^k - 1$ if $n$ is odd, and $2^{k-1}$ if $n$ is even.

Case I(2): $p$ is odd. We have the following result.
Theorem 8 (F, Kwak, Kwon, 2005, 2007)

Let $G$ be a connected circulant graph of order $n$. For any odd prime $p$, the number of isomorphism classes of connected typical circulant $p$-fold coverings of $G$ is $\frac{1}{p-1} (p^{\lfloor \frac{d}{2} \rfloor} - 1)$ if $(p, n) = 1$ and is $p^{\lfloor \frac{d}{2} \rfloor} - 1$ otherwise, where $d$ is the valency of $G$. 
Case II: $r$ is a positive integer.

Lemma 9

The composition of any two typical circulant covering projections is typical. Conversely, for a composite number $r = r_1 r_2$, any typical circulant $r$-fold covering projection is a composition of a typical circulant $r_1$-fold covering projection and a typical circulant $r_2$-fold covering projection.

By using this lemma, we have the following enumeration result for any folding number.
Theorem 10 (F, Kwak, Kwon, 2005, 2007)

Let \( r = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s} \) be the prime factorization of any positive integer \( r \), and let \( G \) be a circulant graph of order \( n \) and valency \( d \). Then the number \( N \) of isomorphism classes of typical circulant connected \( r \)-fold coverings of \( G \) is as follows:

\[
N = \begin{cases} 
0 & \text{if } r \text{ is even and } d \text{ is odd,} \\
\prod_{i=1}^{s} N_i & \text{otherwise,}
\end{cases}
\]

where

\[
N_i = \begin{cases} 
p_i^{r_i (\lfloor \frac{d}{2} \rfloor - 1)} & \text{if } p_i \mid n, \\
p_i^{(r_i - 1)(\lfloor \frac{d}{2} \rfloor - 1)} \left( p_i^{\lfloor \frac{d}{2} \rfloor} - 1 \right) / (p_i^{\lfloor \frac{d}{2} \rfloor} - 1) & \text{if } (p_i, n) = 1.
\end{cases}
\]
Trivalent circulant graphs again I

Corollary 11

Let $G$ be a connected trivalent circulant graph of order $n$ but $G \neq K_4$ or $K_{3,3}$. If $r$ is even, then $G$ has no circulant connected $r$-fold coverings. If $r$ is odd, then $G$ has only one connected circulant $r$-fold covering up to isomorphism.
Trivalent circulant graphs again II

Cases where $G = K_4$ or $G = K_{3,3}$

Theorem 12 (Couperus, 2007)

Let $G = K_4$. If $r$ is even, then $G$ has no circulant connected $r$-fold coverings. If $r$ is odd, then $G$ has 3 connected circulant $r$-fold coverings up to isomorphism, one of them is typical.

Theorem 13 (Couperus, 2007)

Let $G = K_{3,3}$. If $r$ is even, then $G$ has no circulant connected $r$-fold coverings. If $r$ is odd, then $G$ has 6 connected circulant $r$-fold coverings up to isomorphism, one of them is typical.
If we generalize the covering graphs are Cayley graphs on abelian groups (such coverings are called abelian coverings), then we have the following enumeration result.

**Theorem 14 (F, Kwak, Kwon, 2009)**

Let $G = \text{Cay}(\mathbb{Z}_n, Y)$ be a connected circulant graph of order $n$ and valency $d$. Then, for any prime $p$, the number of isomorphism classes of connected typical abelian $p$-fold coverings of $G$ is \( \frac{1}{p-1}(p^\lfloor \frac{d}{2} \rfloor - 1) \).
Thanks