The Gauss-Bonnet Formula of Polytopal Manifolds and the Characterization of Embedded Graphs with Nonnegative Curvature

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Abstract

Let $M$ be a connected $d$-manifold without boundary obtained from a (possibly infinite) collection $\mathcal{P}$ of polytopes of $\mathbb{R}^d$ by identifying them along isometric facets. Let $V(M)$ be the set of vertices of $M$. For each $v \in V(M)$, define the discrete Gaussian curvature $\kappa_M(v)$ as the normal angle-sum with sign, extended over all polytopes having $v$ as a vertex. Our main result is as follows: If the absolute total curvature $\sum_{v \in V(M)} |\kappa_M(v)|$ is finite, then the limiting curvature $\kappa_M(p)$ for every end $p$ of $M$ can be well-defined and holds the Gauss-Bonnet formula:

$$\sum_{v \in V(M) \cup \text{End}M} \kappa_M(v) = \chi(M).$$

In particular, if $G$ is a (possibly infinite) graph embedded in a 2-manifold $M$ without boundary such that every face has at least 3 sides, and if the combinatorial curvature $\Phi_G(v) \geq 0$ for all $v \in V(G)$, then the number of vertices with non-vanishing curvature is finite. Furthermore, if $G$ is finite, then $M$ has four choices: sphere, torus, projective plane, and Klein bottle. If $G$ is infinite, then $M$ has three choices: cylinder without boundary, plane, and projective plane minus one point.